

Original Article

Box-Counting Fractal Dimension of the Lorenz Attractor Across the Route to Chaos: A Bootstrapped, Multi-Estimator Analysis

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Abstract - When describing chaotic attractors, the fractal dimension is a quantitative measure of their geometric complexity, in addition to qualitative visualisation. The box-counting dimension of the Lorenz attractor is estimated for the range of the bifurcation parameter, $\rho \in [0, 30]$, with the following pipeline: occupied-box counting on a fixed absolute grid ladder, 128-replicate grid-origin bootstrap for confidence interval, and a two-segment piecewise-linear fit for automatic selection of the scaling band. In the steady regime ($\rho \lesssim 22$), the box counting dimension is nearly zero, and it increases rapidly to about 1.8 near the onset of chaos ($\rho \approx 23$), and thus the box counting dimension serves as a numerical measure of the onset of chaos. At the classical parameter set ($\sigma = 10$, $\rho = 28$, $\beta = 8/3$), the box-counting dimension is 1.86 (95% confidence interval 1.84–1.87) and increases monotonically towards the theoretical value with sample size. The estimate is compared with the Grassberger–Procaccia correlation dimension (2.03) and the Kaplan–Yorke dimension based on the Lyapunov spectrum (2.06). It is found that the capacity dimension is systematically underestimated by box-counting when computed on finite trajectories and that the dimension of a strange attractor is sensitively dependent on the estimator used: a walking-divider dimension gives a dimension around unity, consistent with a trajectory itself being regarded as a curve, not the attractor set.

Keywords - Box-counting dimension, Chaos, Fractal dimension, Kaplan–Yorke dimension, Lorenz attractor.

1. Introduction

A fractal is a geometric shape that is self-similar, meaning it repeats itself in many different ways; the complexity of the fractal can be described by a fractal dimension [2, 3]. The geometry of strange attractors created by deterministic chaotic systems is fractal, and the fractal dimension is a compact and quantitative description of the geometry of the attractor [15, 16].

The Lorenz system was proposed in 1963 as a simplified version of atmospheric convection [1] and is the prototype example of deterministic chaos. The trajectories converge to a strange attractor in the shape of a butterfly for the classical parameters, $\sigma = 10$, $\rho = 28$, and $\beta = 8/3$ (as depicted in Figure 1). Most treatments in the beginning focus on three dimensional visualisation of this attractor, but do not quantify its geometry.

Box counting is a general and widely used technique for computing the fractal dimension of any point set [8, 11]. Its basic manifestation is Mandelbrot's well-known research on coastlines [2] and has been recently used for Indian coastlines

with explicit uncertainty quantification [18]. In principle, the same geometric tool could be used for the point cloud of a chaotic trajectory.

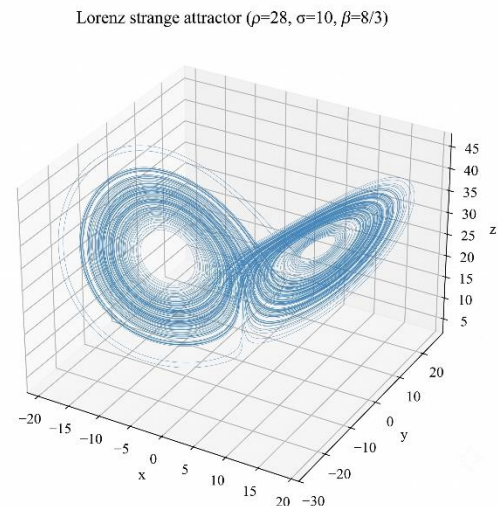


Fig. 1 The Lorenz strange attractor at $\sigma = 10$, $\rho = 28$, $\beta = 8/3$



In this paper, the box-counting dimension of the Lorenz attractor is estimated as a function of the bifurcation parameter ρ , and it is questioned whether the box-counting dimension could be used as a numerical measure of the onset of chaos. The estimate at $\rho = 28$ is checked with the theoretical Kaplan–Yorke dimension (≈ 2.06) and the Grassberger–Procaccia correlation dimension. Uncertainty quantification and automatic scaling-band selection based on coastline analysis remove the subjectivity of the box-size range: a grid-origin bootstrap is adapted.

2. Materials and Methods

2.1. The Lorenz System and the Numerical Integration

The Lorenz system is given by Equation 1.

$$dx/dt = \sigma(y - x), \quad dy/dt = x(\rho - z) - y, \quad dz/dt = xy - \beta z \quad (1)$$

The equations were solved using the explicit Runge–Kutta 5(4) method (the `solve_ivp` routine of SciPy [12]) with relative and absolute tolerances of 10^{-9} . $\sigma = 10$ and $\beta = 8/3$ were held fixed while ρ was varied. The first 40–60 time units of each trajectory were discarded as transient, so that the retained points lie on the attractor — or on the settled fixed point or limit cycle for non-chaotic ρ .

2.2. Box-Counting Dimension

The point cloud consisting of the trajectory points is enclosed in a cubic grid of side ε , and the number of boxes filled with at least one point is enumerated as $N(\varepsilon)$. If the number of occupied boxes in the fractal set scales according to Equation 2, then the dimension D is the slope of the $\log N(\varepsilon)$ vs. $\log(1/\varepsilon)$ plot for the fractal set.

$$N(\varepsilon) \propto \varepsilon^{-D}, \quad D = \lim_{\varepsilon \rightarrow 0} [\log N(\varepsilon) / \log(1/\varepsilon)] \quad (2)$$

For the boxes occupied, integer grid indices were assigned to each point, and the number of distinct triples of indices were counted. To compare the dimensions for different values of ρ , all the box sizes were measured using the same absolute scale, which was determined by the spatial size of the canonical chaotic attractor. A fixed point is then covered by essentially one box at every scale, so $N(\varepsilon)$ does not grow, and it correctly returns $D \approx 0$.

2.3. Grid-Origin Bootstrap

A grid-origin bootstrap was used for each box size to minimize sensitivity to the location of the grid origin, and counts were recomputed for 128 replications. The resulting slopes are used to give a 95% confidence interval for D [11].

2.4. Scaling-Band Selection

The finite trajectories do not trace out a single line on the log–log plot; counts saturate at large scales and roll off at small scales because boxes have too few points at the latter scales. The length of the scaling band was determined by fitting a piecewise linear model to the log–log data, and the

slope of the longer segment was taken as D_{band} , the box-counting dimension. For each step, $\Delta\rho = 0.5$, ρ was calculated, and the onset of chaos was determined by the minimum ρ value for which D_{band} was greater than unity.

2.5. Validation Estimators

Three independent estimators were used for validation at $\rho = 28$. A Theiler window was used to remove the scaling of pairs of values that were temporally correlated, and the Grassberger–Procaccia correlation dimension D_2 was estimated. The whole Lyapunov spectrum was calculated according to the method of Benettin and reorthonormalisation of the tangent vectors by the periodic Gram–Schmidt method [10], and the Kaplan–Yorke dimension was extracted from the spectrum using Equation 3 [6, 15].

$$D_{KY} = k + (\lambda_1 + \dots + \lambda_k) / |\lambda_{k+1}| \quad (3)$$

Finally, the time-ordered trajectory was used as a second geometric estimator by applying the walking-divider (Richardson) method [2]. All calculations have been done in Python with the help of the NumPy library [13], the SciPy library [12], and the pwlf library [14]. The full code and generated data have been placed into the supplementary repository, and all the random draws were seeded.

3. Results and Discussion

3.1. Dimension Across the Route to Chaos

The behavior of D_{band} with respect to ρ is plotted in Figure 2. For the steady regime ($\rho \lesssim 22$), the trajectory tends to a fixed point in the steady state, and the dimension is zero. There is a sudden increase in dimension near $\rho = 23$, and then it levels off at about 1.77 for $\rho \geq 24$, where the strange attractor is found. For $D > 1$, the smallest ρ for which this happens is 23.0, very close to the classical value $\rho \approx 24.06$, at which the Lorenz attractor is born, and just below that of the Hopf bifurcation at $\rho = 24.74$ [7]. The box-counting dimension is therefore a clean numerical measure of the onset of chaos. The transitional values at $\rho = 22$ and $\rho = 22.5$, with confidence intervals containing zero, refer to long chaotic transients which have not yet stabilized in the integration time.

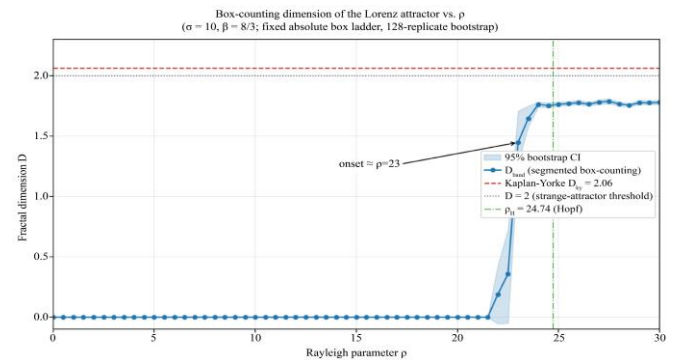


Fig. 2 Box-counting dimension D_{band} versus ρ ($\sigma = 10$, $\beta = 8/3$). Shaded band: 95% bootstrap confidence interval

3.2. Box-Counting at the Classical Parameters

At the classical parameters, the segmented fit yields $D_{\text{band}} = 1.857$, with a 95% confidence interval of [1.845, 1.874] over a scaling band of 5.9 octaves ($R^2 = 0.9999$), as shown in Figure 3. The characteristic roll-off at the finest scales, where boxes begin to contain single points, is visible and is correctly excluded by the segmented fit.

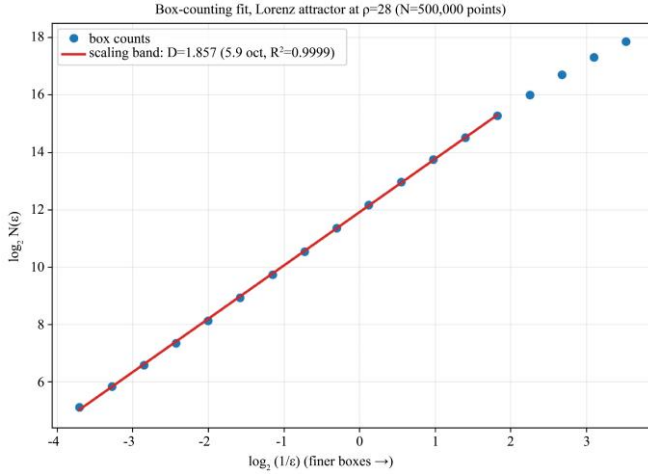


Fig. 3 Log-log box-counting fit at $\rho = 28$ ($N = 5 \times 10^5$ points). The red line is the selected scaling band

3.3. Convergence with Sample Size

Box-counting dimension estimated on a finite trajectory underestimates the true capacity dimension. Figure 4 shows that D_{band} increases monotonically with the number of trajectory points, from 1.71 at 4×10^4 points to 1.89 at 1.6×10^6 points, approaching but not reaching the theoretical value of 2.06 within the sampling used.

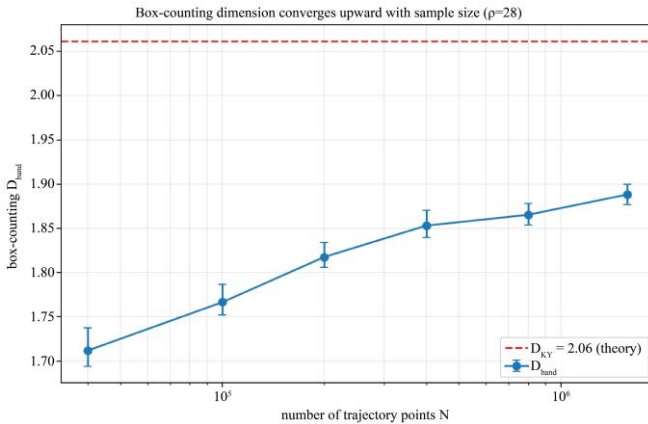


Fig. 4 Box-counting dimension versus number of trajectory points at $\rho = 28$; the dashed line marks the Kaplan–Yorke value 2.06

3.4. Sensitivity to the Box-Size Range

The estimate is robust to the choice of box-size range. Across five different ladders, the dimension varies only between 1.833 and 1.841, a spread of 0.008 (Figure 5), confirming that the segmented-band selection removes the subjectivity usually associated with box-counting.

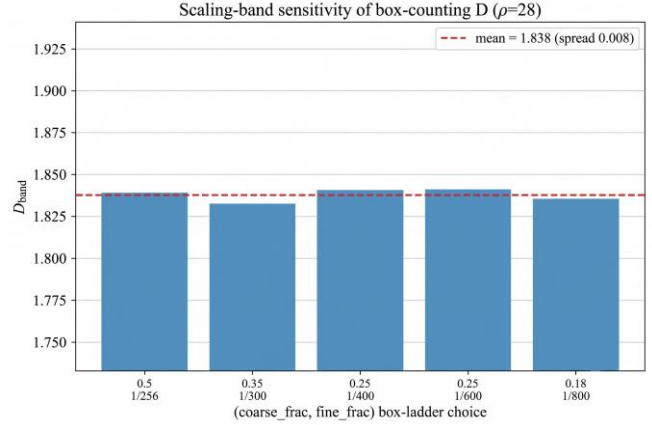


Fig. 5 Box-counting dimension across five box-ladder choices at $\rho = 28$; the spread is 0.008

3.5. Comparison of Estimators

Table 1 and Figure 8 compare the four estimators at $\rho = 28$. The walking-divider method returns $D = 0.93$ (Figure 7): applied to a single orbit, it measures the trajectory as a one-dimensional curve and does not resolve the stacked-sheet structure transverse to the flow. Box-counting gives 1.86, the Grassberger–Procaccia correlation dimension gives 2.03 (Figure 6), and the Kaplan–Yorke dimension computed from the Lyapunov spectrum (0.907, 0.000, -14.57) gives 2.062, in close agreement with the literature value of ≈ 2.06 [15, 16].

Table 1. Dimension of the Lorenz attractor at $\rho = 28$ by four estimators

Estimator	Dimension	What it measures
Walking-divider (Richardson)	0.93	Orbit as a 1-D curve
Box-counting (this work)	1.86	Attractor space-filling
Correlation (Grassberger–Procaccia)	2.03	Pair-density scaling
Kaplan–Yorke (Lyapunov)	2.06	Dynamics (theory)

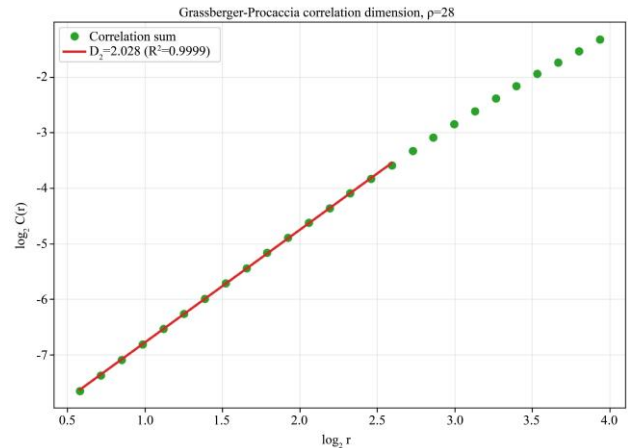


Fig. 6 Grassberger–Procaccia correlation-sum fit at $\rho = 28$ ($D_2 = 2.03$)

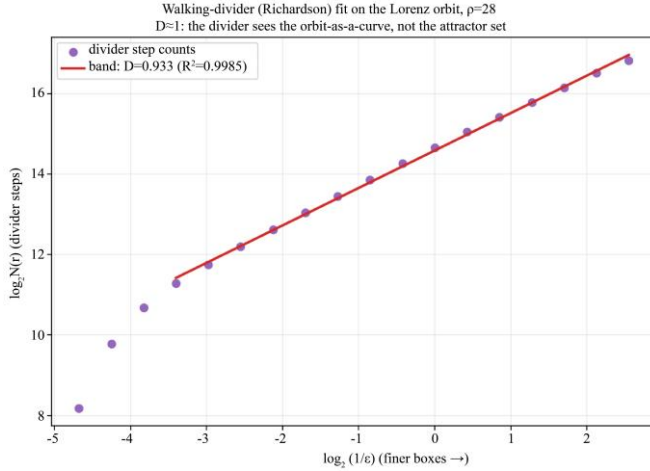


Fig. 7 Walking-divider (Richardson) fit on the $\rho = 28$ orbit ($D \approx 0.93$)

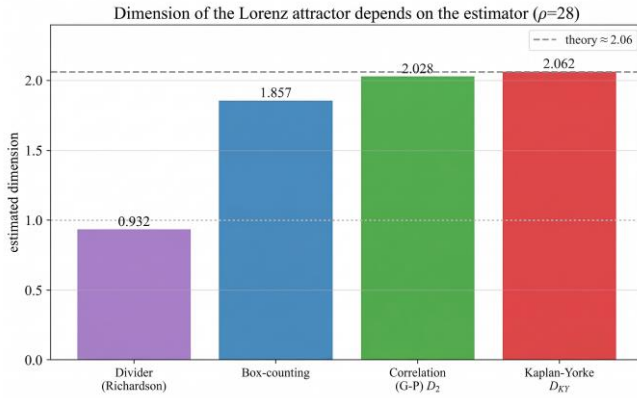


Fig. 8 The four estimators form an ordered ladder; the dashed line is the theoretical value ≈ 2.06

The estimators are therefore placed in an ordered sequence. The systematic differences show that the capacity dimension of a strange attractor is underestimated on finite data, while the correlation dimension and the Kaplan–Yorke

dimension recover the accepted value. This is similar to the result obtained for natural coastlines, where box-counting and divider estimators for the same coastline are systematically different [18].

4. Conclusion

A pipeline from coastline fractal analysis was adapted for bootstrapping the Lorenz attractor dimension as it was tracked from the periodic to the chaotic window, across the route to chaos. We see that in the steady regime the dimension is zero, and around the chaos-onset value of $\rho = 23$ the dimension rises to about 1.8, showing that box-counting dimension is a viable numerical chaos-onset indicator. The box-counting estimate (1.86) is an underestimate of the theoretical capacity dimension and approaches the value from below as the sample size increases, while the correlation dimension (2.03) and the Kaplan–Yorke dimension (2.06) recover the accepted value. The reported dimension of the strange attractor is therefore a function of the estimator. Future research is required to deal with the entire multifractal spectrum, to use the pipeline for other chaotic systems, and to describe the finite-sample bias of the box-counting estimator in the presence of observational noise.

Conflicts of Interest

The author declares that there is no conflict of interest regarding the publication of this paper.

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