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Original Article

Feasibility Study on Use of Nonlinear Film Electrodynamic Waveguide Structure to Control Antenna-Radiated Pulse Shape

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***Abstract** - The theoretical basis for describing both the process of electromagnetic radiation passing through a nonlinear film and its reflection from a thin-film coating with nonlinear properties is considered. The process of generating field harmonics at frequencies determined by the film properties and the characteristics of the incident signal is considered. This method of high-frequency correction of the signal spectrum, using a nonlinear film electrodynamic structure, will allow changing the shape of the pulse, which is transmitted along the waveguide from the generator. Thus, the proposed theoretical basis will determine the possibilities of controlling the shape of pulses that are transmitted along the waveguides without additional energy losses. It is established that the decrease in the duration of the pulse fronts leads to an increase in the energy of high-frequency components in the spectrum of the radiated signal, which allows an increase in the efficiency of focusing the field of most antenna systems. To realize such energy transfer to high-frequency components of the spectrum, it is suggested to add elements with nonlinear characteristics to the antenna.*

***Keywords** - Electromagnetic waveguide, Nonlinear film antenna, Ultra-wideband communication, Soliton, Reflector.*

1. Introduction

In recent years, much attention has been paid to nonlinear waves in various media [1, 2], the properties of which can be used in the creation of new devices. In this case, as a rule, due to the complexity of the theoretical analysis of multidimensional systems, nonlinear waves in one-dimensional systems are considered. Within the framework of electrodynamics problems, real structures of various frequency ranges have a three-dimensional geometry, and only for some types of waves can they be modeled by well-known [3] flat, two-dimensional models.

This significantly complicates the theoretical analysis of such systems and limits the possibilities of creating new electrodynamic devices that include elements with nonlinear characteristics. Waveguide systems with nonlinear films have a number of unique features compared to conventional linear waveguides and, in particular, allow the existence of nonlinear waves (solitons). The spectrum of intrinsic excitations of nonlinear waveguides can contain both solitons of the envelope of electromagnetic waves and stationary video pulses. Stable soliton solutions in nonlinear waveguides are formed by compensating for dispersion distortion of the signal by nonlinearity, which makes it possible to develop high-speed information transmission systems based on nonlinear waveguide structures. The parameters of solitons depend both on the properties of the nonlinear element and on the dispersion characteristics of the waveguide structure.

Modern studies [1, 5, 6] have shown that the properties of solitons of film waveguide structures in microwave frequencies have not been sufficiently studied, and the theory of soliton control of the shape of the emitted pulse of Ultra-Wideband (UWB) antennas has not been considered.

Consequently, it is necessary to develop theoretical foundations for constructing mathematical models for the formation of soliton pulse signals in nonlinear waveguide structures, as well as to determine ways to use such signals in antennas of various types effectively.Therefore, it is relevant to develop the theoretical foundations for the construction of thin-film nonlinear waveguide microwave systems, including UWB communication systems with controlled characteristics of pulsed radiation.

2. Literature Analysis and Problem Statement

In [1-7], a soliton is defined as a structurally stable individual wave that propagates in a nonlinear delta and behaves like a particle (particle-like wave), which, when interacting with each other or with some other perturbations, does not collapse but continues to move, keeping its structure stable. It is proposed in [4-7] to utilize this property to transmit data over long distances without interference.

However, in [4], it is pointed out that in multidimensional conditions, mathematically describing the stability of solitons, which is maintained by spatial modulation of nonlinearity, is a challenging problem for both theoretical and experimental studies.

To study the problem of describing "soliton control" of wave propagation processes in [5, 6], it is proposed to use quartz matrices doped with ions of rare-earth elements, including the recording of single and coherent systems of optical waveguides in their thickness. In [6], it is proposed to build a model of the process of formation of solitons of ultra-short duration, taking into account the dependences of material parameters of the medium on the spectral component of the passing pulse to use a partial solution of the nonlinear Schrödinger Equation. Thus, in [5, 6], the possibility of forming stationary solitons of the field, whose parameters can be controlled by an external wave of optical pumping, was theoretically proved.

However, due to the specific features of soliton propagation in different media, a separate, unique solution to the problem of controlling the shape of the emitted (re-emitted) soliton(s) in film microwave waveguides and reflector antennas is required. However, there are difficulties in describing the theoretical basis of "soliton signal control" in current-film waveguide and antenna systems, such as:

- A complex mathematical description of the boundary conditions for the polarized electromagnetic field on the surface of a film with a stepped nonlinearity;
- No mathematical model exists to describe the process of forming solitons of the electromagnetic wave envelope in waveguide structures, including the thin dielectric films with a diagonal tensor of dielectric permittivity with step nonlinearity;
- Mathematical dependence of soliton parameters on the nonlinear film parameters and dispersion characteristics of the microwave structure (waveguides, antennas, etc.) has not been established to enable control of the shape of the soliton signal.

To overcome these difficulties in this study, the theoretical basis for the electromagnetic analysis of soliton passage through thin-film waveguides and on the surface of antennas has been developed. On the basis of the proposed theoretical provisions, mathematical relations for calculating the parameters of solitons of the generalized structure of waveguide thin-film systems can be obtained, and the dependence of the parameters of such solitons on the parameters of the nonlinear film can be established.

The mathematical models proposed here will provide for the creation of devices for controlling the shape of the signal on the basis of nonlinear waveguides and, in particular, will reduce the duration (sharpening) of the output pulse signal ("soliton signal control").

It is necessary to mathematically describe not only the process of creating solitons in films but also to consider methods of controlling the parameters (width, amplitude) of pulses in different waveguide elements and structures (fiber guides, antennas).

2.1. Purpose and Objectives of the Study

This work aims to theoretically describe the processes of excitation of soliton pulse signals in nonlinear waveguide thin-film structures, as well as to determine the possibility of effective control of radiation of such signals by antennas of various types with film coating. To achieve the study's goal, it is necessary to solve the following tasks:

- Perform the analysis of boundary conditions for the electromagnetic field on the surface of a nonlinear film;
- Carry out spectral analysis of soliton perturbations and build a model of processes of excitation of soliton pulse signals in nonlinear waveguide structures, taking into account thin-film technologies;
- Analyze the properties of broadband nonlinear excitations in a strip line and determine ways to control the parameters (duration and amplitude) of the radiation pulse by nonlinear antennas of various types with the use of solitons;
- Determine possible ways to control the shape of the transmitted signal in a waveguide with thin-film walls or radiated by antennas with a film coating of the reflector.

3. Materials and Methods

3.1. Boundary Conditions for the Electromagnetic Field on the Surface of Thin Films with Nonlinear Characteristics

Let us define the boundary conditions for the electromagnetic field on the surface of a thin film with nonlinear properties. Accounting for the influence of thin layers of different materials on the electrodynamic parameters of various structures due to the smallness of the thickness parameter kd (d – is the thickness of the layer, $k=2\pi/\lambda$, λ – wavelength), is effectively carried out by using equivalent boundary conditions [1, 8], which significantly simplify the analysis of structures. Consider a thin ($kd \ll 1$) layer of nonlinear dielectric located in the yz interface plane of two media with linear parameters (Figure 1).

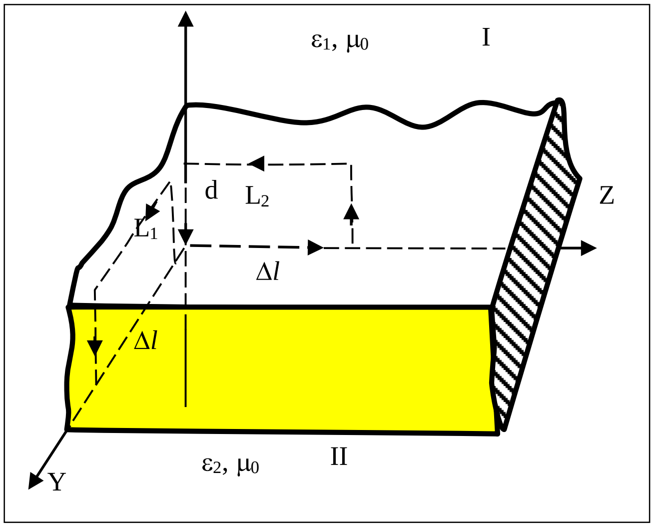

Fig. 1 Toward the derivation of boundary conditions on a nonlinear film of thickness d

Application of Maxwell's Equations in the integral form of recording

$$\oint Edl = -\frac{d}{dt}\iint_S BdS,$$

$$\oint Hdl = -\frac{d}{dt}\iint_S DdS,$$

Where $B=\mu H+M_{NL}$, $D=\varepsilon\varepsilon_0 E+P_{NL}$, M_{NL}, P_{NL} – vectors, in the general case, of nonlinear magnetization and polarization to the contours *L1* and *LII* (Figure 1) leads to ratios:

$$E_z\big|_{x=0}\Delta l + E_x\big|_{z=\Delta l} d - E_z\big|_{z=d}\Delta l - E_x\big|_{z=0} d = -\Delta l d\frac{dB_y}{dt} + O(kd)^2,$$

$$E_y\big|_{x=d}\Delta l + E_x\big|_{y=0} d - E_y\big|_{z=0}\Delta l - E_x\big|_{y=\Delta} d = -\Delta l d\frac{dB_z}{dt} + O(kd)^2,$$

$$H_z\big|_{x=0}\Delta l + H_x\big|_{z=\Delta l} d - H_z\big|_{x=d}\Delta l - H_x\big|_{z=0} d = -\Delta l d\frac{dD_y}{dt} + O(kd)^2,$$

$$H_y\big|_{x=d}\Delta l + H_x\big|_{y=0} d - H_y\big|_{x=0}\Delta l - H_x\big|_{y=\Delta l} d = -\Delta l d\frac{dD_z}{dt} + O(kd)^2,$$

Where $O(kd)^2$ – values of the second order of smallness. Considering the continuity of the tangential components E_y, E_z, H_y, H_z and normal components B_x, D_x of the fields in the interface planes $x=0$, $x=d$, and assuming that inside the thin layer:

$$E_{y,z} = \frac{1}{2}\left(E_{y,z}^{(1)} + E_{y,z}^{(2)}\right), \quad H_{y,z} = \frac{1}{2}\left(H_{y,z}^{(1)} + H_{y,z}^{(2)}\right),$$

E_y, E_z, H_y, and H_z were obtained for the monochromatic components of the total field:

$$\begin{bmatrix} E_z^{(2)} - E_z^{(1)} + \frac{1}{2}\frac{d}{\bar{\varepsilon}}\frac{d}{dz}(\varepsilon_1 E_x^{(1)} + \varepsilon_2 E_x^{(2)}) = -i\omega\mu_0\frac{d}{2}(H_y^{(1)} + H_y^{(2)}) - i\omega dM_{NL_y} \\ E_y^{(2)} - E_y^{(1)} + \frac{1}{2}\frac{d}{\bar{\varepsilon}}\frac{d}{dy}(\varepsilon_1 E_x^{(1)} + \varepsilon_2 E_x^{(2)}) = i\omega\mu_0\frac{d}{2}(H_z^{(1)} + H_z^{(2)}) + i\omega dM_{NLz} \\ H_z^{(2)} - H_z^{(1)} + \frac{d}{2}\frac{d}{dz}(H_x^{(1)} + H_x^{(2)}) = i\omega\varepsilon_0\bar{\varepsilon}\frac{d}{2}(E_y^{(1)} + E_y^{(2)}) + i\omega dP_{NL_y} \\ H_y^{(2)} - H_y^{(1)} + \frac{d}{2}\frac{d}{dz}(H_x^{(1)} + H_x^{(2)}) = -i\omega\varepsilon_0\bar{\varepsilon}\frac{d}{2}(E_z^{(1)} + E_z^{(2)}) + i\omega dP_{NLz} \end{bmatrix}. \quad (1)$$

Without specifying the kind of nonlinear polarization and magnetization, which in the linear case pass into the known relations [10].

3.2. Solitons of the Electromagnetic Wave Envelope in Waveguide Structures with Nonlinear Films

Let us now consider the spectrum of eigen excitations and the possibility of the existence of soliton solutions on the example of a regular waveguide structure (e.g., a rectangular waveguide with a multilayer dielectric) with a nonlinear film in the *yz* plane of the media interface. Such a structure can be represented for generality as a three-layer system: a thin nonlinear layer (Figure 1), described by (1), located between layered half-spaces I and II with linear parameters characterized by tensors of input impedances, known for a wide range of specific isotropic and anisotropic structures [13].

In the absence of anisotropy in the yz plane, the tensors are diagonal; assuming d/dz=0, it is easy to see that independent propagation of $E(E_x, E_y, E_z)$ and $H(H_x, H_y, H_z)$ - waves is possible in the structure. Let us consider the H-waves. Based on the definition of the input impedance, it can be written down the relations in the planes x=0 x=d, respectively,

$$E_z^{(1,2)} = Z_{1,2}H_y^{(1,2)} = i\omega\mu_0 Z_{1,2}H_y^{(1,2)} \quad (2)$$

For the nonlinear film with parameters $\mu, \varepsilon_{zz} = \varepsilon + k|E|^2$, $(M_{NL} = 0)$after the transition to Fourier components, the boundary conditions take form.

$$\begin{aligned} & E_z^{(2)} - E_z^{(1)} = -i\omega\mu_0\frac{d}{2}(H_y^{(1)} + H_y^{(2)}) \\ & H_y^{(2)} - H_y^{(1)} + \frac{d}{2}\frac{d}{dy}(H_x^{(1)} + H_x^{(2)}) = \\ & = -i\omega\varepsilon_0\bar{\varepsilon}\frac{d}{2}(E_z^{(1)} + E_z^{(2)}) - i\omega dP_{NL_z} \end{aligned} \quad (3)$$

Based on Maxwell's Equations and Equations 2 and 3, was obtained the following equation in terms of $E_z(\omega, k)$:

$$F(\omega, k)\,E_z(\omega, k) = -P_{NL}(\omega, k) \quad (4)$$

Where,

$$F(\omega,k)=\frac{1}{\omega^2\varepsilon_0\mu_0 d}\left[2d\left(\frac{Y_2}{1+\alpha^{-1}}-\frac{Y_1}{1+\alpha}\right)+k^2\right]-\bar{\varepsilon}$$
$$\alpha=(1-Y_1\frac{d}{2})(1+Y_2\frac{d}{2})^{-1}, \quad (5)$$
$$Y_1=Z_1^{-1},\ Y_2=Z_2^{-1},\ \frac{d}{dy}\to k.$$

In the plane y, t - representation Equation 4 has the form:

$$\iint_{-\infty}^{+\infty}\frac{dk d\omega}{(2\pi)^2}\,F(\omega,k)E_z(\omega,k)e^{-i(ky-\omega t)}=-P_{NL}(y,t) \quad (6)$$

Moreover, describes the waveguiding properties of the structure at arbitrary nonlinearity $P_{NL}(y, t)$. The solution of the equation depends on the nonlinear term P_{NL}. It will look for the solution of the equation in the form of momentum

$$E_z(y,t)=e(y,t)e^{[-i(Qy-\Omega t)} \quad (7)$$

Satisfying the slowness condition.

$$\left|\frac{de(y,t)}{dt}\right|\ll\Omega|e(y,t)|,\left|\frac{de(y,t)}{dy}\right|\ll Q|e(y,t)| \quad (8)$$

This means that the components are essentially different from zero only in the region of,

$$|k-Q|<\tau_s^{-1} \quad (9)$$

Where τ_s – pulse duration, v – pulse velocity.

Assuming that at the spectral width of the pulse, i.e., in the region Equation 9, the function *F(ω, k)* changes slowly enough and its features lie outside this region when calculating the left part of Equation 6, then the decomposition will be used taking into account the spatial dispersion:

$$F(\omega,k)=F(\Omega,Q)+\left.\frac{dF}{d\omega}\right|_{\substack{\omega=\Omega\\k=Q}}(\omega-\Omega)+$$
$$+\left.\frac{dF}{dk}\right|_{\substack{\omega=\Omega\\k=Q}}(k-Q)+\frac{1}{2}\left.\frac{d^2F}{d^2\omega}\right|_{\substack{\omega=\Omega\\k=Q}}(\omega-\Omega)^2+\left.\frac{d^2F}{d\omega dk}\right|_{\substack{\omega=\Omega\\k=Q}}(\omega-\Omega)(k-Q) \quad (10)$$
$$+\frac{1}{2}\left.\frac{d^2F}{d^2k}\right|_{\substack{\omega=\Omega\\k=Q}}(k-Q)^2+\ldots\ .$$

Substituting Equation 10 into 6, taking into account the decomposition terms written out and integrating taking into account the properties of Fourier transforms leads to the equation for the function e(y, t):

$$F(\Omega,Q)e(y,t)+i\left.\left\{\frac{dF}{d\omega}\frac{d}{dt}-i\frac{dF}{dk}\frac{d}{dy}\right\}\right|_{\substack{\omega=\Omega\\k=Q}}e(y,t)+ \quad (11)$$
$$+\left.\left\{-\frac{1}{2}\frac{d^2F}{d\omega^2}\frac{d^2}{dt^2}+\frac{d^2F}{d\omega dk}\frac{d^2}{dydt}-\frac{1}{2}\frac{d^2F}{dk^2}\frac{d^2}{dy^2}\right\}\right|_{\substack{\omega=\Omega\\k=Q}}e(y,t).$$

Which is a generalization of the nonlinear Schrodinger equation. If the relation,

$$\propto\left(\frac{d^2F}{d\omega^2}+\frac{1}{v^2}\frac{d^2F}{dk^2}+\frac{2}{v}\frac{d^2F}{d\omega dk}\right)|_{\substack{\omega=\Omega\\k=Q}}<0 \quad (12)$$

Then, the solution of Equation 6, as it can be checked directly by substitution, is the function

$$e(y,t)=E_s cn(z,n) \quad (13)$$

Where E_s is the soliton amplitude, cn(z, n) is the elliptic Jacobi function, n∈(0,1)– is the modulus, $z=(t-y/v)\tau_s$-1, $v=d\Omega/dQ$.

Substituting Equation 13 into 1 gives formulas for determining the duration of solitons.

$$\frac{1}{\tau_s^2}=\frac{\propto E_s^2}{n^2}\left(\frac{d^2F}{d\omega^2}+\frac{2}{v}\frac{d^2F}{d\omega dk}+\frac{1}{v^2}\frac{d^2F}{dk^2}\right)^{-1}|_{\substack{\omega=\Omega\\k=Q}} \quad (14)$$

The ratio between Q and Ω.

$$\Omega\left.\left(\frac{d^2F}{d\omega^2}+\frac{1}{v}\frac{d^2F}{d\omega dk}\right)\right|_{\substack{\omega=\Omega\\k=Q}}+Q\left.\left(\frac{d^2F}{d\omega dk}+\frac{1}{v}\frac{d^2F}{dk}\right)\right|_{\substack{\omega=\Omega\\k=Q}}+ \quad (15)$$
$$+\left.\left(\frac{dF}{d\omega}+\frac{1}{v}\frac{dF}{dk}\right)\right|_{\substack{\omega=\Omega\\k=Q}}=0,$$

And the dispersion equation,

$$\left[F(\Omega,Q)+\Omega\frac{dF}{d\omega}+Q\frac{dF}{dk}+\frac{1}{2}\frac{d^2F}{d\omega^2}\left(\Omega^2+\frac{1-2n^2}{\tau_s^2}\right)+\right. \quad (16)$$
$$\left.\left.+\frac{1}{2}\frac{d^2F}{dk^2}\left(Q^2+\frac{1-2n^2}{(v\tau_s)^2}\right)+\frac{d^2F}{d\omega dk}\left(\Omega Q+\frac{1-2n^2}{(v\tau_s)^2}\right)\right]\right|_{\substack{\omega=\Omega\\k=Q}}=0\ ,$$

or

$$\left.\left[\begin{matrix}F(\Omega,Q)+\Omega\frac{dF}{d\omega}+Q\frac{dF}{dk}+\frac{1}{2}\frac{d^2F}{d\omega^2}\Omega^2+\\+\frac{1}{2}\frac{d^2F}{dk^2}Q^2+\frac{d^2F}{d\omega dk}\Omega Q\end{matrix}\right]\right|_{\substack{\omega=\Omega\\k=Q}}+ \quad (17)$$
$$+\frac{2n^2-1}{2n^2}\propto E_s^2=0.$$

In particular, at *n=1*, have a single envelope soliton

$$E_z(y,t) = E_s \sec h\left(\frac{t-y/v}{\tau_s}\right) e^{[i(Qy-\Omega t)]} \quad (18)$$

If the relation,

$$\aleph\left(\frac{d^2F}{d\omega^2} + \frac{1}{v^2}\frac{d^2F}{dk^2} + \frac{2}{v}\frac{d^2F}{d\omega dk}\right)\Big|_{\substack{\omega=\Omega \\ k=Q}} > 0 \quad (19)$$

The solution (6) has the form,

$$e(y,t) = E_s sn(z,n) \quad (20)$$

$$e(y,t) = E_s sn(z,n=1) = E_s th\left[\frac{\left(t-\frac{y}{v}\right)}{\tau_s}\right] \quad (21)$$

Herewith,

$$\frac{1}{\tau_s^2} = \frac{\aleph E_s^2}{n^2}\left(\frac{d^2F}{d\omega^2} + \frac{1}{v^2}\frac{d^2F}{dk^2} + \frac{2}{v}\frac{d^2F}{d\omega dk}\right)\Big|_{\substack{\omega=\Omega \\ k=Q}} > 0 \quad (22)$$

The relation between Q and Ω does not change, and the dispersion equation has the following form,

$$\left[F(\Omega,Q) + \Omega\frac{dF}{d\omega} + Q\frac{dF}{dk} + \frac{1}{2}\frac{d^2F}{d\omega^2}\Omega^2 + \frac{1}{2}\frac{d^2F}{dk^2}Q^2 + \frac{d^2F}{d\omega dk}\Omega Q\right]\Bigg|_{\substack{\omega=\Omega \\ k=Q}} - \quad (23)$$

$$-\frac{1+n^2}{2n^2}\aleph E_s^2 = 0 \quad .$$

At n=1 have the case of the "darkening" soliton envelope.

The analysis of the obtained relations shows that the propagation of solitons is possible in a regular waveguiding structure with nonlinear films, the character of which is determined by the sign of nonlinearity and dispersion characteristics of the structure. In the absence of a nonlinear film (d→0, ε→0, k→0) the dispersion equation goes to the well-known relation for a linear structure F(ω, k)=Y1(ω, k)-Y2(ω, k)=0 due to both the linear part of the dielectric permittivity of the film and the nonlinear part proportional to the signal level E_s^2.

The soliton parameters depend significantly on both the film parameter k, the signal level E_s^2, and the steepness of the characteristics F(ω,k) of the frequency and wave number functions E_s^2. Thus, by modulating the parameters of the linear part of the waveguide structure, e.g., by sub-magnetizing the introduced ferrite layers, the soliton parameters can be controlled. The obtained relations allow modeling different designs of waveguide systems based on the required soliton parameters for the available films with nonlinear parameters. A similar analysis is performed for *E*-waves. However, the soliton parameters of *E*-waves and *H*-waves in the same structure are different because the input impedances of such waves are different. In the presence of anisotropy in the *yz* plane and are tensors and, *E*-waves and *H*-waves are interconnected in the anisotropic region, the study of nonlinear waves in such structures is of independent interest.

3.3. A Method for Analyzing the Properties of Broadband Nonlinear Excitations in a Film with Nonlinear Parameters Soliton Solutions in Microstrip Lines

The processes of envelope soliton formation in waveguides with nonlinear parameters considered in the previous section may be of practical interest in the development of noise-resistant communication systems with increased information transmission rates. [7, 8] Waveguide nonlinear systems also allow the existence of broadband pulse excitations - video solitons. Broadband pulses can be generated, for example, with the help of size-quantum films of semiconductors, taking into account the peculiarities of their dispersion characteristics. It is considered a method for analyzing the properties of broadband nonlinear excitations and studying the properties of such excitations in a waveguide structure, including a film with nonlinear parameters.

As a nonlinear waveguide, it is chosen a strip line including dielectrics with linear field parameters ε_2, μ_0, and ε_1, μ_0, and a film between them described, in general, by the nonlinear polarization vector $P_n(E)$. Let us consider the peculiarities of wave propagation in such a strip structure. The boundary conditions in the plane of the film are as follows

$$E_{z1}\left(x=\frac{\delta}{2}\right) = E_{z2}\left(x=\frac{\delta}{2}\right)$$
$$H_{y1}\left(x=\frac{\delta}{2}\right) - H_{y2}\left(x=-\frac{\delta}{2}\right) = \frac{e\delta dE_z}{cdt} + \frac{4\pi\delta dP_n(E)}{cdt} \quad (24)$$

Where δ – film thickness; ε – linear part of the dielectric permittivity of the film.

The solution is sought in the form of the eigenfunction expansion of the linear part of the structure.

$$E_{zi}(x,y,t) = \iint \frac{d\omega dk}{(2\pi)^2} E_{zi}(\omega,k)\varphi_i(x)\exp[i(\omega t - ky)] \quad (25)$$

Substituting Equation 4 into 2 gives the relation between E(y, t) and E(ω, k) (x∈0),

$$\int\int_{-\infty}^{\infty} \frac{d\omega dk}{(2\pi)^2} Y(\omega,k)Y(\omega,k)\exp[i(\omega t - ky)] = \frac{e\delta}{c}\frac{d^2E}{dt} + \frac{4\pi\delta}{c^2}\frac{P_n(E)}{dt^2} \quad (26)$$

Equation 26, together with the fourier transform of *E(y, t)*, gives a nonlinear integro-differential equation with respect to the field distribution function in the film region with nonlinear parameters.

Equation 5 is very general and describes various waveguide structures. The function Y(ω, k) is determined by the boundary conditions, in particular, in a strip structure using an acceptable magnetic wall model at low frequencies.

$$\left| \begin{array}{l} H_{y1}(x=r) = H_{2y}(x=-d) = 0 \\ Y(\omega,k)p\sin(pr) + qtg(qd)\cos(pr) \end{array} \right. \quad (27)$$

Where transverse wave numbers,

$$p^2 = \frac{\varepsilon_1\omega^2}{c^2} - k^2;\; q^2 = \frac{\varepsilon_2\omega^2}{c^2} - k^2$$

To obtain analytical solutions to Equation 5 it is used the decomposition.

$$\begin{aligned} Y(\omega,k) &= p\left[pr - \frac{(pr)^3}{3!} + \frac{(pr)^5}{5!} + \ldots\right] + \\ &+ q\left[qd + \frac{(qd)^3}{3} + \frac{2(qd)^5}{15} + \ldots\right]\times \\ &\times\left[1 - \frac{(pr)^2}{2!} + \frac{(pr)^4}{4!} + \ldots\right]. \end{aligned} \quad (28)$$

In this case, Equation 5 reduces to the nonlinear equation.

$$\begin{aligned} &\left[\begin{array}{l} \alpha_1\frac{d^2}{dt^2} + \alpha_2\frac{d^2}{dy^2} + \alpha_3\frac{d^4}{dt^4} + \alpha_4\frac{d^4}{dt^2dy^2} + \\ \alpha_5\frac{d^4}{dy^4} + +s_1\frac{d^6}{dt^6} + s_2\frac{d^6}{dt^4dy^2} + s_3\frac{d^6}{dt^2dy^4} + \ldots \end{array}\right]\times \\ &\times E(y,t) = \frac{e\delta}{c}\frac{d^2E}{dt^2} + \frac{4\pi\delta}{c^2}\frac{d^2P_n(E)}{dt^2}, \end{aligned} \quad (29)$$

Where,

$$\begin{aligned} &\alpha_1 = (e_1r + e_2d)c^{-2};\; \alpha_2 = r + d; \\ &\alpha_3 = \left(\frac{e_1^2r^3}{3!} + \frac{e_1e_2r^2d}{2} - \frac{e_2^2d^2}{3}\right)c^{-4}; \\ &\alpha_4 = \left(\frac{2e_2d^3}{3} - \frac{(e_1+e_2)r^2d}{2} - \right). \end{aligned}$$

The solution of the equation describing the evolution of pulses in a structure with a nonlinear film depends on the type of nonlinearity. Here are the formulas for calculating the parameters of single stationary pulses at the most typical types of nonlinearity. At quadratic nonlinearity $P_n(E)=\chi E^2$ pulses of the type,

$$E(y,t) = E_s ch^{-2}[\tau_s^{-1}(t - y/v)] \quad (30)$$

Where E_s – amplitude; τ_s – duration; v – pulse velocity:

$$v = \left\{\frac{\alpha_2}{\alpha_1 + \frac{\delta}{\delta/n^2(e+(8\pi/3)\chi E_s)}}\right\}^{\frac{1}{2}};$$

$$\tau_s^{-1} = \left\{\frac{2\pi\delta\chi E_s}{c^2(\alpha_3 + \alpha_4v^{-2} + \alpha_5v^{-4})}\right\}^{\frac{1}{2}}.$$

In the more general case of nonlinearities $P_n(E)=\chi E^2+\chi E^3$ the character of the pulse does not change, and its parameters do not change:

$$\begin{aligned} &v = \left\{\frac{\alpha_2}{\alpha_1 + \frac{\delta}{\delta/n^2(e+(8\pi/3)\chi E_s)}}\right\}^{\frac{1}{2}} \\ &\tau_s^{-1} = \left\{\frac{2\pi\delta\chi E_s}{c^2(\alpha_3 + \alpha_4v^{-2} + \alpha_5v^{-4})}\right\}^{\frac{1}{2}}. \end{aligned} \quad (31)$$

In the case of cubic nonlinearity $P_n(E)=\chi E^3$ along with Equation 6, the solution, depending on the excitation, can have the form,

$$E(y,t) = E_s ch^{-1}\left[\tau_s^{-1}(t - y/v)\right] \quad (32)$$

Parameterized,

$$\left.\begin{array}{l} v = \left\{\frac{\alpha_2}{\alpha_1 + \frac{\delta}{c^2}(e + 2\pi\gamma E_s^2)}\right\}^{\frac{1}{2}}; \\ \tau_s^{-1} = \left\{\frac{2\pi\delta\gamma E_s^2}{c^2(\alpha_3 + \alpha_4v^{-2} + \alpha_5v^{-4})}\right\}^{\frac{1}{2}} \end{array}\right\}. \quad (33)$$

In a constant electric field E_0 in the case of cubic nonlinearity, it is possible to excite "dimming" pulses $E(y, t)=th\{\tau_s^{-1}(t-y/v)\}$ parameterized.

$$\left.\begin{aligned} v &= \left\{ \frac{\alpha_2}{\alpha_1 + \frac{\delta}{c^2}(e + 2\pi\gamma E_s^2)} \right\}^{\frac{1}{2}}; \\ \tau_s^{-1} &= \left\{ \frac{-2\pi\delta\gamma E_s^2}{c^2(\alpha_3 + \alpha_4 v^{-2} + \alpha_5 v^{-4})} \right\}^{\frac{1}{2}} \end{aligned}\right\} \quad (34)$$

The pulse duration and propagation speed are determined by the pulse amplitude E_s, which is characteristic of nonlinear waves. Stationary pulses are formed due to the joint action of nonlinearity and dispersion, which leads to a qualitative change in the characteristics of waveguide structures and provides new opportunities in the technique of information transmission and processing:

1. Compensation of dispersion distortions of signals allows for an increase in the speed of information transmission.
2. High stability of pulses - reduce the requirements to the degree of inhomogeneity of the path.
3. Interaction of broadband pulses with inhomogeneous parts of the structure allows one to realize on their basis broadband generators of coherent radiation.
4. The phenomenon of bi-stability or multistability observed at higher orders of nonlinearity allows us to realize new devices of microwave range.

At excitation in the structure of a periodic signal, its transformation into a periodic lattice of pulses described by elliptic functions with a periodic in time phenomenon of return to the initial state occurs. The described method can be used to calculate waveguide structures of various types. Higher types of nonlinearity (4th order, etc.) are similarly taken into account, but the relations for determining the parameters of pulses represent a system of equations requiring numerical investigation.

3.4. Peculiarities of the Passage of Pulse Signals through Films with Nonlinear Parameters

In this section it is considered the peculiarities of the passage of electromagnetic pulse signals through films with nonlinear properties. The passage of electromagnetic waves through interfaces is well investigated in the linear approximation.

Due to the complexity of the analysis, the interaction of electromagnetic radiation with nonlinear boundaries is considered mainly in the analysis of harmonic generation and frequency interactions of monochromatic signals. The nonlinear nature of the boundary conditions makes it impossible to directly use the Fourier transform method to calculate the pulse shape by the nature of the passage of each spectral component separately.

Therefore, the calculations are performed in the time domain and the pulse shape is determined as a result of solving the corresponding nonlinear equation. Here, it is considered a method for calculating pulse parameters using both frequency and time domain methods of analysis. The parameters of pulses are studied when they pass through the interface in free space or a waveguide with a nonlinear film oriented perpendicular to its axis. It has been established that at influences in the form of step functions, an approximate analytical solution of integrodifferential nonlinear equations describing the passage of pulses through a film with nonlinear parameters is possible.

Let us consider, for simplicity a planar model of the structure (Figure 2), which models well the main types of waveguides (strip, coaxial, etc.). The wave equations in the regions of the structure with linear parameters 1 and 2 have the following form

$$\left(-\varepsilon_{1,2}\mu_{1,2}\frac{\omega^2}{c^2}\right)E(x,y,z) = 0 \quad (35)$$

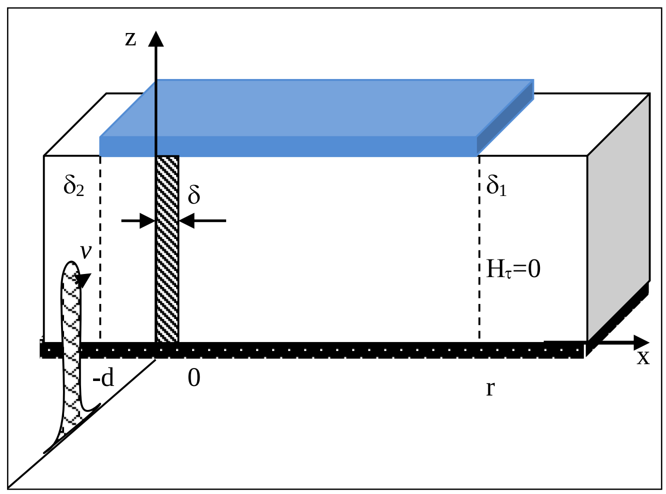

Fig. 2 Microstrip line with waveguide film

Boundary conditions in the interface plane

$$E_{1\tau}(y=0-) - E_{2\tau}(y=0+) = 0 \quad (36)$$

$$H_{1\tau}(y=0-) - H_{2\tau}(y=0+) = \frac{d}{dt}(\mathrm{n} * P_n(\mathrm{E})) \quad (37)$$

Where the nonlinear polarization vector is represented as,

$$P_n(E) = \sum_{n=2}^{\infty} k_n E_n^n \quad (38)$$

For certainty, let us consider $H(H_x, H_y, H_z)$ waves (d/dz=0). The solution of (2) is sought in the form of decompositions.

$$E_{O,R,T}(x,y,z) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \Phi(x) E_{O,R,T}(\omega) e^{[i(\omega t - k_{1,2}y)]} \quad (39)$$

Where the indices O, R, and T correspond to the incident, reflected, and transmitted impulses through the interface. Substitution of (6) into (4) gives the relations between the spectral components, and substitution into (5) after multiplication by the transverse field distribution function ϕ(x) and integration over the cross-section of the structure gives an equation that reduces to a nonlinear integrodifferential equation with respect to the field distribution function with a nonlinear function included in the differential operator. For the main wave of the strip and coaxial wave ϕ(x)≈1. With a small dispersion of media in regions I and II defined by the functions $\varepsilon_i(\omega)$, $\mu_i(\omega)$ for the main wave, it is obtained a nonlinear ordinary differential equation.

$$\alpha \sum_{n=2}^{\infty} k_n E_T^{n-1}(y=0,t)\frac{dE_T(y=0,t)}{dt} + E_T(y=0,t) = F_0(t) \tag{40}$$

Where

$$\alpha = 4\pi\delta c^{-1}(\mu_2/\varepsilon_2)^{\frac{1}{2}}, F_0(t) = E_0(y,t)(\varepsilon_1\mu_2/\varepsilon_2\mu_1)^{1/2}$$

In the steady-state regime (at $dE_0/dt=0$) from (7) follows,

$$E_T(t) = (\varepsilon_1\mu_2/\varepsilon_2\mu_1)^{\frac{1}{2}} E_0(t) \tag{41}$$

Which corresponds to the linear theory. Thus, the film with nonlinear parameters affects the transients during the pulse passage. Let the impact have the form of a rectangular pulse:

$$F_0(t) = \begin{cases} F_0, & 0 < t < T \\ 0, & t < 0, t > T \end{cases} \tag{42}$$

Then, in the region $0 < t < T$, the solution (6) has the form:

$$E(t) = -\alpha \sum_{n=2}^{\infty} k_n n \Big\{ C_{n-1}^0 F_0^{n-1} \ln \frac{F_0 - E_T(t)}{F_0 - E_T(0)} - C_{n-1}^1 F_0^{n-2}[E_T(0) - E_T(t)] + \\ + \frac{1}{2} C_{n-1}^2 F_0^{n-3}[(F_0 - E_T(t))^2 - (F_0 - E_T(0)]^2 + \ldots + (-1)^{n-1} C_{n-1}^{n-1} F_0 \times \\ \times [(F_0 - E_T(t))^{n-1} - (F_0 - E_T(0))^{n-1}] \frac{1}{n-1} \Big\},$$

Where $E_T(t) -$ is the initial value of the function at the moment of the influence pulse arrival $C_n^m = n!/[m!(n-m)!]$. In the area of $t>T$ function $E_T(t)$ is defined as,

$$E_T(t) = T + \sum_{n=2}^{\infty} \frac{k_n n}{n-1} [E_T^{n-1}(T) - E_T^{n-1}(t)] \tag{43}$$

In particular, in the case of quadratic nonlinearity:

$$\begin{vmatrix} E(t) = 2\alpha k_2 \left[E_r(t) - E_r(0) + F_0 1n \frac{F_0 - E_r(t)}{F_0 - E_r(0)}\right] & 0 < t < T \\ E_r(t) = E_r(T) - 2\alpha k_2 t \ \ t > T \end{vmatrix} \tag{44}$$

The degree of nonlinearity affects the steepness of the pulse edge and decline. The approximation of the input pulse $E_0(t)$ as a set of step functions allows us to use expressions (8) and (9) to determine the shape of the pulses passed through the film at arbitrary shapes of pulses $E_0(t)$. Films with other types of nonlinearity can be similarly considered. The case of higher wave types requires a separate description because of the interaction of modes in the region of the film with nonlinear parameters. Thus, along with the systems of stationary pulse formation in waveguide structures [1, 2, 13], films with nonlinear properties can be used in devices for the formation of pulse signals.

3.5. Reflection of Electromagnetic Radiation from a Conductive Surface with a Nonlinear Film Coating

Let us consider the peculiarities of the reflection of a pulse signal from a metallic surface with a coating having nonlinear characteristics. The problem in this formulation is of interest in the study of the possibility of controlling the shape of the pulse signal through the use of a nonlinear coating of the conductive surface of the antenna reflector. Let us also analyze regularities of change of the form of the pulse signal at the reflection from coverings with nonlinear dielectric and magnetic permeabilities. When deriving the boundary conditions for the electric and magnetic fields on an ideally conducting surface with a nonlinear coating (Figure 3) should take into account that on one of the sides of the film (on the surface of the metal), the condition $E_R=0$.

Suppose that an electromagnetic signal containing the field components E_z H_y in a vacuum falls on the surface under consideration. Let us also assume that the spectral components of the signal incident on the film satisfy the condition kd<<1, i.e., the film thickness is small compared to the characteristic wavelength of the pulse. Then the boundary conditions follow from Maxwell's equations with accuracy up to higher powers of the small parameters d/λ and taking into account $E_z|_{x=0}=0$.

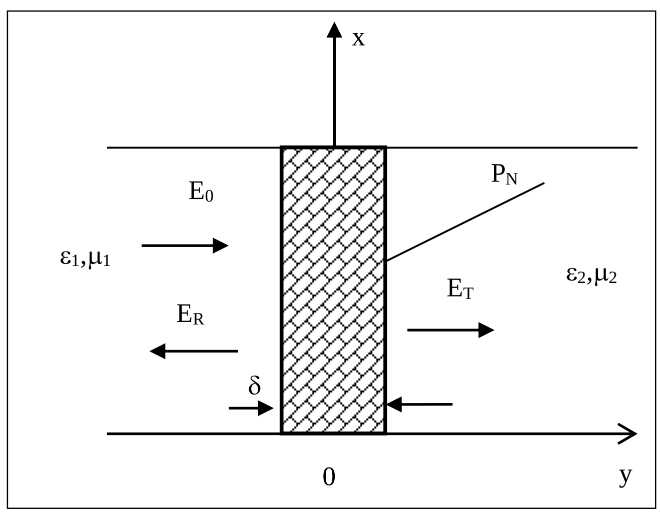

Fig. 3 Two-dimensional model of a waveguide structure oriented perpendicular to the waveguide axis

$$(E_z)|_{x=d} = -d\frac{\partial \tilde{B}_y}{\partial t} + 0((kd)^2) \quad (45)$$

$$H_y|_{x=d} - H_y|_{x=0} = d\frac{\tilde{D}_z}{dt} + 0((kd)^2) \quad (46)$$

Where $\tilde{B}, \tilde{D}$ - electric and magnetic permeabilities in the nonlinear film. The nonlinear character of the boundary conditions (2)-(3) corresponds to the presence of nonlinear field additives in the expressions for electric and magnetic induction.

$$\tilde{B} = \mu\tilde{B} = \bar{\mu}\tilde{H} + M_{NL} \quad (47)$$

$$\tilde{D} = \mu\tilde{E} = \bar{e}\tilde{E} + P_{NL} \quad (48)$$

Let us denote the electromagnetic field components corresponding to the signal incident on the surface by (E_z^0, H_y^0), to the reflected signal through (E_z^-, H_y^-) and the passed signal (on a metal surface) - through the ($E_z^+=0$, H_y^+).

Then, the boundary conditions (4) will be rewritten in the form:

$$E_z^0 + E_z^- = -d\frac{d\tilde{B}_y}{dt} \quad (49)$$

$$H_y^0 + H_y^- = -d\frac{d\tilde{B}_y}{dt} \quad (50)$$

Consider first the reflection of a signal from a coated metal surface whose dielectric permittivity is a nonlinear function of the field strength, and there is no nonlinearity in the expression for the magnetic permeability ($M_{NL}=0$). It will be assumed that the following relation determines the dielectric properties of the film:

$$\varepsilon = \bar{\varepsilon} + \square|\mathrm{E}|^2 \quad (51)$$

In this case, using the relations between the components of the electric and magnetic fields of the signal in vacuum $E_z^0=ZB_z^0$, $E_z^-=-ZB_z^-$ ($Z=\sqrt{(\mu/\varepsilon)}$ – impedance of the medium) and substituting the expressions for the electric and magnetic fields in the film it is obtained:

$$\tilde{B}_y = \frac{1}{2}\left(B_y|_{x=0} + B_y|_{x=d}\right) \quad (52)$$

$$\tilde{E}_y = \frac{1}{2}\left(E_y|_{x=0} + E_y|_{x=d}\right) \quad (53)$$

Obtained the equations for determining the shape of the reflected signal E_z^- by the shape of a given incident signal E_z^0

$$\frac{d}{2z}\frac{dE_z^-}{dt} - \frac{d}{2}\frac{dB_y^+}{dt} = \frac{d}{2}\frac{dB_y^0}{dt} + E_z^- + E_z^0 \quad (54)$$

$$\frac{d}{4}\square\frac{d(E_z|E_Z|^2)}{dt} + \frac{d}{2}\varepsilon\frac{dE_z}{dt} = B_y^0 - \frac{1}{z}E_z^- - B_y^+ \quad (55)$$

Where $E_z=E_z^0+E_z^-$.

Expressing from the second equation B_y^- through the components of electric field strength, it is obtained the equation for the field E_z:

$$\frac{d^2}{dt^2}\left(1+\frac{\aleph}{2\bar{\varepsilon}}E_z^2\right)E_z - \frac{4}{\bar{\varepsilon}Zddt}E_z - -\frac{4}{\bar{\varepsilon}d^2}E_z = \frac{8}{\bar{\varepsilon}Zdt}E_z^0. \quad (56)$$

As can be seen from (56), when using a coating with an electrical character of nonlinearity in the linear d/λ approximation, the nonlinearity does not affect the shape of the reflected signal. This is due to the fact that due to the boundary condition (2), the value of the electric field strength in the film is proportional to the film thickness. The shape of the reflected signal can be determined from E_Q.

$$E_z^- = -E_z^0 - \frac{2d}{Z}\frac{\overline{\mu d}}{dt}E_z^0 \quad (57)$$

In the considered approximation, the shape of the reflected signal will not differ from the shape of the incident signal. The additive present proportional to the film thickness corresponds to the shift of the reflected signal for a time equal to the time of signal passage through the nonlinear film. Let us now consider the peculiarities of the reflection of pulse signals from a metal surface with a coating whose magnetic permeability is a nonlinear function of the magnetic field strength.

$$\mu = \bar{\mu} + \square|\mathrm{B}|^2 \quad (58)$$

Using the boundary conditions and performing calculations similar to those given above, it is obtained an expression for the electric field strength of the signal reflected from a perfectly conducting surface with a magnetic nonlinear coating.

$$E_z^- = -E_z^0 - \frac{2d}{Z}\frac{d}{dt}(\bar{e}E_z^0 + 4\square|E_z^0|^2E_z^0 \quad (59)$$

The change in the waveform described by (59), similar to the previously considered case, can be interpreted as a shift in time. However, now due to nonlinearity, the shift will be different for the signal sections corresponding to different electromagnetic field strengths, which will be the cause of the signal fronts twisting. In this case, depending on the sign of the value of *k*, both the leading and trailing edges of the signal can be twisted.

Figure 4 shows the shape of the reflected signal depending on the amplitude of the incident signal. The shape of the incident signal was chosen to be Gaussian and is described by the expression.

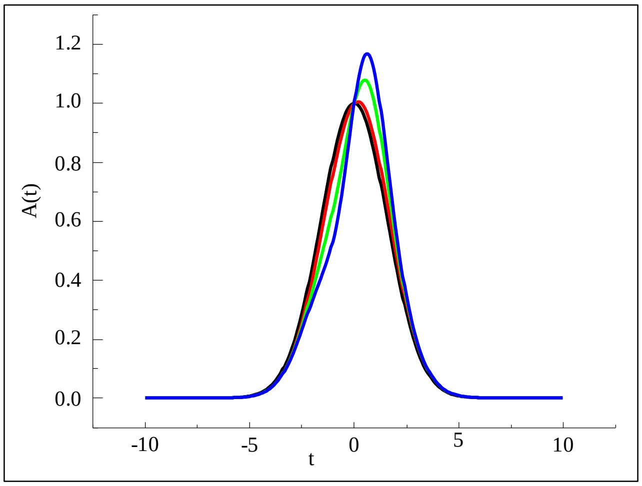

Fig. 4 Process of sharpening of the pulse signal reflected from the surface of the metal with nonlinear coating with increasing nonlinearity parameter

$$E_z^0(t)|_{x=0} = Ae^{-(t-t_0)^2} \qquad (60)$$

As can be seen from (Figure 4), the curvature of the signal edge increases with increasing amplitude of the incident signal. Thus, the above results show the possibility of using nonlinear coatings of antenna system elements to form signals with a given shape.

4. Results and Discussion

The results of the development of theoretical bases of electromagnetic signal shape control in nonlinear waveguides suggest there are both advantages and disadvantages. The advantages indicated by the conducted research are:

1. The boundary conditions (1), (3), (24), (36) for the electromagnetic field on the surface of a thin film with nonlinear properties are formed;
2. Using (10) and (11), a spectral analysis was performed on soliton perturbations in a three-layer structure. This structure has an inner layer that is a nonlinear film. The known input impedance tensors of isotropic and anisotropic structures characterize the outer layers;
3. The theoretical bases for investigation of properties of broadband nonlinear excitations (solitons) in a strip line, including dielectrics with linear field parameters ($\varepsilon_1\mu_0$, $\varepsilon_2\mu_0$,), as well as a film between them, described in general, by the vector of nonlinear polarization $P_n(E)$ (38), are presented;
4. Both frequency and time methods are used to study the parameters of pulses as they pass through the interface between media in free space or a waveguide with a nonlinear film oriented perpendicular to its axis, in accordance with (39) and (41);
5. The regularities, described by (57) and (59), of the pulse signal shape changes during reflection from coatings with nonlinear dielectric and magnetic permeabilities are investigated.

However, there are disadvantages:

a. The given boundary conditions for the electromagnetic field on the surface of a thin film with nonlinear properties do not take into account nonuniformities (surface roughness, defects of the crystal lattice of the film), which is especially necessary to take into account when used in Ultra-High Frequency (UHF) communication systems.
b. When analyzing solitons in waveguide structures, the case of higher wave types requires a separate description because of the interaction of frequency modes in the film region with different types of nonlinearity.
c. The work does not:

- Analyze of amplitude-frequency characteristics of a strip waveguide with a nonlinear film;
- Study the possible degradation of the nonlinear film during its long-term operation in the microwave range;
- Investigate films with all existing types of nonlinearity, taking into account higher wave types;
- Study soliton emergence processes when signals with complex frequency structure (frequency modulation, frequency-phase modulation, etc.) pass through various film nonlinear surfaces.

However, the above disadvantages and technological problems do not limit the possibility of using the considered theoretical basis for the creation of nonlinear soliton structures for controlling the radiation of microwave devices and UHF structures. For example, now promising standards are being developed, such as IEEE 802.15 and, for example, for ultra-wideband communication in the frequency band 7.5 GHz pulse length will be only 150^{-12} s. That is, each short soliton pulse can correspond to a small quantity of information that will greatly simplify the control of signal processing in the receiver.

5. Conclusion

- Boundary conditions for the polarized electromagnetic field on the surface of a nonlinear film have been formulated.
- The principles of spectral analysis of soliton disturbances in a three-layer structure (the inner layer is a nonlinear film, and the outer layers are characterized by tensors of input known impedances of isotropic and anisotropic structures) have been developed.
- The relationships obtained allow the modeling of various designs of waveguide systems based on the required soliton parameters for the available films with nonlinear parameters.

- Analyze the properties of broadband, nonlinear excitations in a strip line, including dielectrics with linear field parameters and a conducting, nonlinear film. It is established that at pulse influences in the form of step functions, an approximate analytical solution of integro-differential, nonlinear equations describing the passage of pulses through a film with nonlinear parameters is possible.
- The developed theoretical foundations allow us to determine the ways of controlling the shape of the pulse signal by using a nonlinear film coating of the conductive surface of the antenna reflector.

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