SSRG International Journal of Electrical and Electronics Engineering
ISSN: 2348-8379/ https://doi.org/10.14445/23488379/IJEEE-V13I9P113

Volume 13 Issue 9, 159-173, September 2026

Original Article

Sensitivity Analysis of Uniform Control Systems

Oleg Gasparyan[1*], Nerses Nersisyan[2], Haykanush Darbinyan[3], Ovanna Ohanyan[4], Mariam Darakhchyan[5], Vahan Manukyan[6], Mkrtich Harutyunyan[7], Davit Danielyan[8]

[1,2,3,4,5,6,7,8]*Control System Chair, National Polytechnic University of Armenia, Yerevan, Armenia.*

[1*]*Correspondent Author : ogasparyan@polytechnic.am*

Received: 06 June 2026 Revised: 08 September 2026 Accepted: 15 September 2026 Published: 26 September 2026

***Abstract** - This study develops a sensitivity framework tailored to uniform multivariable control systems, whose identical dynamic channels are interconnected through a constant numerical coupling matrix. The analysis is carried out using the Characteristic Transfer Functions (CTFs) method. Unlike a general MIMO system, a uniform system contains two structurally distinct sources of parameter variation: changes in the channel transfer functions and changes in the cross-connection matrix. These two cases are examined separately. First-order formulas are obtained for the variations of the CTFs and the canonical basis directions. It is shown that perturbations confined to the identical channel dynamics leave the nominal canonical basis unchanged, whereas perturbations of the coupling matrix generally modify both its eigenvalues and eigenvectors and therefore alter the effective gains and canonical directions of the characteristic systems. Relations connecting the sensitivity functions of the open-loop and closed-loop characteristic systems are also established. The results are illustrated by a sensitivity analysis of a uniform indirect guidance system of a space astronomical space telescope.*

***Keywords** - Multivariable control system, Uniform system, Characteristic Transfer Functions, Canonical basis, Sensitivity, Parameter perturbation.*

1. Introduction

Uniform systems form an important class of Multi-Input, Multi-Output (MIMO) control systems [1, 2]. They consist of dynamically identical channels coupled by fixed cross-connections and arise in applications such as coordinated mechanical and electromechanical devices, distributed actuators, robotic systems, aerospace vehicles, and process control. Their transfer matrix has a particularly transparent structure: the scalar transfer function of an individual channel multiplies a constant matrix that specifies the interconnections between channels. This separation between dynamics and coupling is the defining feature of the class and must be preserved in the analysis and design of any uniform MIMO system.

The paper can be considered as an extension of the research presented in [3] to the sensitivity analysis of uniform systems in the framework of the Characteristic Transfer Functions (CTF) method [4-8]. Generally, the sensitivity analysis examines how system characteristics change when model parameters undergo small variations [9]. In multivariable feedback systems, such variations may affect not only characteristic values but also the directions that define the canonical decomposition of the transfer matrix. Consequently, a complete analysis should describe both the displacement of the characteristic transfer functions and the rotation or deformation of the associated canonical basis axes [1, 2]. The CTFs method provides a convenient input-output framework for this purpose. For an N-dimensional MIMO system, the method associates the transfer matrix with N isolated scalar characteristic systems. Their transfer functions are the eigenvalues of the transfer matrix, while the corresponding right and dual eigenvectors define its canonical basis. This representation makes it possible to interpret many multivariable properties through familiar scalar feedback relations.

Early studies of CTFs sensitivity focused mainly on indices describing the response of the gain loci to perturbations of the return-ratio matrix [10]. Related robustness procedures were later formulated using normalizing pre-compensators and commutative controllers [11]. These approaches provide useful theoretical insight, but they do not directly exploit the specific two-block structure of uniform systems.

This research is not based on a routine substitution into general formulas derived in [3]. The transfer matrix of a uniform system contains two physically and mathematically different components: the common channel transfer function and the numerical cross-connection matrix. Perturbations of these components have different consequences. Variations of identical channel dynamics scale the characteristic systems without changing the nominal eigenvectors of the coupling matrix. By contrast, perturbations of the cross-connections matrix generally change the eigenvalues and eigenvectors of that matrix; therefore, they affect both the equivalent gains of the CTFs and the orientation of the canonical basis. If the individual channels are perturbed, the uniform structure itself is disturbed, and the resulting CTFs variations must be evaluated accordingly.

The objective of this work is to obtain a sensitivity description that explicitly reflects these structural distinctions. Two groups of perturbations are considered. The first group comprises small variations in the transfer-function parameters of the separate channels. The second group comprises small variations in the entries or parameters of the cross-connection matrix. For each group, first-order expressions are derived for the CTF sensitivities and, where applicable, for the sensitivity vectors of the canonical basis axes. The corresponding relations for the closed-loop characteristic systems are then obtained.

The practical relevance of the results is demonstrated using an indirect guidance (tracking) system of an astronomical telescope mounted on the Earth satellite in a three-axis Cardan gimbal. The example shows how the derived formulas can be used to identify the characteristic channels and canonical directions that are most affected by parameter deviations.

The remainder of the paper is organized as follows. Section 2 recalls the canonical representation of uniform systems and establishes the notation used in the sensitivity derivations. Section 3 treats the two classes of parameter perturbations and develops the first-order sensitivity formulas for the open-loop and closed-loop systems. Section 4 applies the results to the telescope example. The main conclusions are summarized in Section 5.

2. Canonical Representation of Uniform Control Systems

The matrix block diagram of a uniform system is shown in Figure 1 [1, 2].

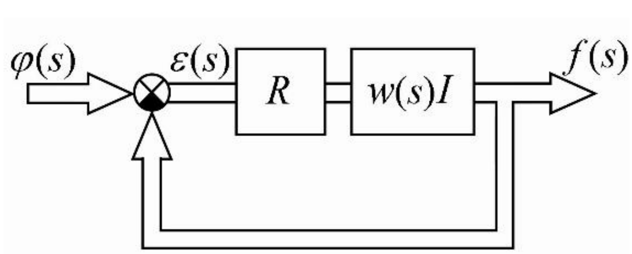


Fig. 1 Matrix block diagram of a uniform MIMO system

Here, $\varphi(s)$, $f(s)$, $\varepsilon(s)$ denote the Laplace transforms of the N-dimensional input, output, and error vector signals $\varphi(t)$, $f(t)$, $\varepsilon(t)$, respectively; these signals are regarded as elements of an N-dimensional complex space $\mathbb{C}^N$. The quantity $w(s)$ is the scalar transfer function of individual channels and is assumed to be a proper rational function of the complex variable s. The matrix R is a numerical matrix of order $N \times N$ describing the cross-connections between the channels.

It follows immediately that the transfer matrix $W(s)$ of the open-loop uniform system shown in Figure 1,

$$W(s) = w(s)R \tag{1}$$

coincides, up to the complex scalar multiplier $w(s)$, with the numerical cross-connection matrix R. This structural property gives rise to several important dynamic and geometric features of uniform MIMO systems and justifies treating them as a distinct class of multivariable control systems [1, 2].

The output vector$f(s)$ and the error vector $\varepsilon(s)$,

Where

$$\varepsilon(s) = \varphi(s) - f(s),$$

are related to the input vector $\varphi(s)$ through the operator equations:

$$f(s) = T(s)\varphi(s), \varepsilon(s) = S(s)\varphi(s) \tag{2}$$

Where,

$$T(s) = [I + w(s)R]^{-1}w(s)R \tag{3}$$

and

$$S(s) = [I + w(s)R]^{-1} \tag{4}$$

are the closed-loop transfer function matrices with respect to the output and error signals, and I is the unit matrix. The matrices $T(s)$ (3) and $S(s)$ (4) are commonly referred to as the complementary sensitivity function matrix and the sensitivity function matrix, respectively [1, 2].

Using the method of CTFs, the transfer matrix of the open-loop uniform system $W(s)$ (1) can be represented, by means of dyadic notation and similarity transformation, in the following canonical forms [4-6]:

$$W(s) = \sum_{i=1}^{N} c_i > q_i(s) < c_i^+ = C diag\{q_i(s)\}C^{-1} \tag{5}$$

Where the complex scalar functions

$$q_i(s) = \lambda_i w(s), i = 1,2,\dots,N \tag{6}$$

are the CTFs of the open-loop system. For simplicity, all

λ_i in (6) are assumed to be distinct. The vectors c_i are linearly independent, normalized eigenvectors of R and constitute the canonical basis of the open-loop uniform system. The vectors c_1^+ are dual to c_i, and the constant modal matrix C is formed by the column-vectors c_i.

Substituting the canonical representations of $W(s)$ in (5) into the complementary sensitivity matrix $T(s)$ (3) and the sensitivity matrix $S(s)$ (4) yields [4]

$$T(s) = \sum_{i=1}^{N} c_i > \frac{\lambda_i w(s)}{1 + \lambda_i w(s)} < c_i^+ = Cdiag\left\{\frac{\lambda_i w(s)}{1 + \lambda_i w(s)}\right\} C^{-1} \quad (7)$$

$$S(s) = \sum_{i=1}^{N} c_i > \frac{1}{1 + \lambda_i w(s)} < c_i^+ = Cdiag\left\{\frac{1}{1 + \lambda_i w(s)}\right\} C^{-1} \quad (8)$$

Expressions (5), (7) and (8) imply that the canonical basis of the closed-loop uniform system coincides with that of the open-loop system [4]. Moreover, the CTFs of the closed-loop MIMO system with respect to both the output and error are related to $q_i(s)$ by the same relationships that connect the open-loop and closed-loop transfer functions in common SISO control systems [9].

The stability of the uniform system shown in Figure 1 is determined by the roots of the characteristic equation [1, 2]

$$det\ [I + w(s)R] = \prod_{i=1}^{N} [1 + \lambda_i w(s)] = 0$$

It follows directly from this equation that the uniform system is stable if all N SISO characteristic systems are stable. Therefore, the stability analysis of an N -dimensional uniform system reduces to the stability analysis of N independent characteristic systems. This analysis may be carried out using standard methods of classical feedback control, for example, the Nyquist criterion [9].

3. Sensitivity of Characteristic Transfer Functions and Canonical Basis Axes of Uniform Systems

This section examines the influence of small parameter variations on the dynamic and structural properties of uniform systems. In general, such problems are addressed by specialized methods of sensitivity theory [12-16].

Here, the sensitivity of uniform systems is analyzed within the framework of the CTFs method. Particular attention is devoted to the sensitivity functions of the SISO characteristic subsystems and of the canonical basis axes under small perturbations of the system parameters.

Assume that the open-loop transfer matrix $W(s)$ of the uniform system shown in Figure 1 depends continuously on m varying parameters α_r $(r = 1, 2, \ldots, m)$ collected in the vector α. To emphasize this dependence on α, the transfer matrix is denoted by $W(s, \alpha)$. Let α_o represents the vector of nominal parameter values, and let $\Delta\alpha$ denotes the corresponding perturbation vector, so that $\alpha = \alpha_o + \Delta\alpha$. Suppose that the CTFs and canonical basis associated with the nominal parameter vector α_o are known.

The problem is then to determine how these CTFs and the canonical basis axes change when the parameters are slightly perturbed, that is, for $(\Delta\alpha_r)^2 \approx 0$. In what follows, this problem is treated in the first approximation by applying perturbation theory for linear operators in finite-dimensional Hilbert spaces [17, 18]. Because of the specific structure of uniform control systems, it is natural to distinguish between two separate sensitivity problems.

The first concerns the sensitivity of the uniform system to perturbations of the *m* parameters α_r $(r = 1, 2, \ldots, m)$ of the transfer functions $w(s)$. In this case, the scalar transfer matrix $w(s, \alpha)I$ is replaced by a diagonal matrix $diag\{w(s, \alpha_i)\}$, in which each *m*-dimensional vector α_i has the form $\alpha_i = \alpha_o + \Delta\alpha_i$. Evidently, for the unperturbed system (if all $\Delta\alpha_i$ are zero). we have $diag\{w(s, \alpha_i)\} = w(s, \alpha_0)I$. The second problem concerns the case in which the varying parameters are elements of the numerical cross-connections matrix R, i.e., $R = R(\alpha)$, where α is an *m*-dimensional vector. In both cases, it is assumed that the transfer functions $w(s, \alpha_i)$ or the matrix $R(\alpha)$ depend continuously on the parameters α_{ir} or α_r.

If the system parameters are perturbed, the transfer matrix $W(s, \alpha_o)$ can be represented as

$$W(s, \alpha) = W(s, \alpha_o) + \Delta W(s, \alpha) \quad (9)$$

Where $\Delta W(s, \alpha)$ denotes the variation of the transfer matrix $W(s, \alpha_o)$ induced by the perturbations $\Delta\alpha$.

3.1. Open-Loop uniform system

Let us begin with the sensitivity analysis of the CTFs and the canonical basis axes of the open-loop system.

3.1.1. Sensitivity of the CTFs to Small Perturbations of the Transfer Functions $w(s, \alpha_i)$

For each *i*, we have

$$w(s, \alpha_i) = w(s, \alpha_0) + \Delta w(s, \alpha_i) \quad (10)$$

$i = 1, 2, \ldots, N,$

and the transfer matrix $W(s,\alpha)$ in (9) takes the form:

$$W(s,\alpha) = W(s,\alpha_o) + \Delta W(s,\alpha) = diag\{w(s,\alpha_i)\}R = w(s,\alpha_o)R + diag\{\Delta w_i(s,\alpha_i)\}R \quad (11)$$

Where

$$W(s,\alpha_o) = w(s,\alpha_o)R,$$
$$\Delta W(s,\alpha) = diag\{\Delta w(s,\alpha_i)\}R \quad (12)$$

and α_i is an m -dimensional vector with the components α_{ir} $(r = 1,2,\ldots,m)$.

Since the transfer functions $w(s,\alpha_i)$ are assumed to depend continuously on α_i, the variation $\Delta W(s,\alpha)$ in (12) can be expanded for small perturbations $\Delta\alpha_{ir}$ by means of the Taylor series

$$\Delta W(s,\alpha) = diag\{\textstyle\sum_{r=1}^{m} U_{ir}^{W}(s)\Delta\alpha_{ir} + \ldots\}R \quad (13)$$

Where

$$U_{ir}^{W}(s) = \frac{\delta w(s,\alpha_i)}{\delta\alpha_{ir}}\bigg|_{\alpha_i = \alpha_o} \quad (14)$$

$$i = 1,2,\ldots,N; \quad r = 1, 2, \ldots, m,$$

are the first-order sensitivity functions of the transfer function $w(s,\alpha_i)$ with respect to parameter variations $\Delta\alpha_{ir}$, evaluated at the nominal vector $\alpha_i = \alpha_o$. The dots in (13), and in what follows mean that the terms of the higher order of smallness, as compared with $\Delta\alpha_{ir}$, are omitted.

Since the transfer functions $w(s,\alpha_i)$ are differentiable with respect to α_{ir}, the CTFs $q_i(s,\alpha)$ admit the expansions:

$$q_i(s,\alpha) = q_i(s,\alpha_o) + \textstyle\sum_{r=1}^{m} S_{ir}^{q}(s)\Delta\alpha_{ir} + \ldots \quad (15)$$

$$i = 1,2,\ldots,N,$$

Where

$$S_{ir}^{q}(s) = \frac{\delta q_i(s,\alpha_i)}{\delta\alpha_{ir}}\bigg|_{\alpha_i = \alpha_o} = \lambda_i \frac{\delta w(s,\alpha_i)}{\delta\alpha_{ir}}\bigg|_{\alpha_i = \alpha_o} \quad (16)$$

$$r = 1, 2, \ldots, m$$

are the unknown first-order sensitivity functions of the CTFs $q_i(s,\alpha_i)$.

Since the quantities in (15) are the "eigenvalues" of the perturbed matrix $W(s,\alpha)$ (11), and taking (10) into account, one obtains

$$[w(s,\alpha_o)R + diag\{\textstyle\sum_{r=1}^{m} U_{ir}^{W}(s)\Delta\alpha_{ir} + \ldots\}R]c_i = diag\{q_i(s,\alpha_o) + \textstyle\sum_{r=1}^{m} S_{ir}^{q}(s)\Delta\alpha_{ir} + \ldots\}c_i \quad (17)$$

$$i = 1, 2, \ldots, N$$

Using the corresponding first-order approximations for $Rc_i = \lambda_i c_i$ and $q_i(s,\alpha_o) = \lambda_i w(s,\alpha_o)$, immediately yields

$$U_{ir}^{W}(s) = S_{ir}^{q}(s) = \lambda_i \frac{\delta w(s,\alpha_i)}{\delta\alpha_{ir}}\bigg|_{\alpha_i = \alpha_o} \quad (18)$$

$$r = 1, 2, \ldots, m,$$

Thus, the sensitivity functions of the i th characteristic system of the open-loop uniform system with respect to non-identical variations of parameters of the transfer functions $w(s,\alpha_i)$ are equal to the corresponding sensitivities of $w(s,\alpha_i)$ multiplied by the eigenvalue λ_i.

3.1.2. Sensitivity of the Open-Loop Uniform System to Small Perturbations of the Cross-Connection Matrix R

Consider next the case in which the varying parameters belong to the numerical cross-connections matrix R. In this case, the matrix R can be written as,

$$R = R(\alpha) = R(\alpha_o) + \Delta R(\alpha) \quad (19)$$

Assuming continuous dependence of $R(\alpha)$ on α_r, the variation $\Delta R(\alpha)$ in (19) can be expressed as:

$$\Delta R(\alpha) = \textstyle\sum_{r=1}^{m} U_r^{R}\Delta\alpha_r + \ldots, \quad (20)$$

Where

$$U_r^{R} = \frac{\delta R(\alpha)}{\delta\alpha_r}\bigg|_{\alpha = \alpha_o} \quad (21)$$

$$r = 1, 2, \ldots, m$$

are the sensitivity matrices of $R(\alpha)$ with respect to corresponding perturbations $\Delta\alpha_r$. Introduce the following expansions for the canonical basis axes c_i (eigenvectors of R) and factors λ_i (eigenvalues of R):

$$c_i(\alpha) = c_i(\alpha_o) + \textstyle\sum_{r=1}^{m} \Upsilon_{ir}\Delta\alpha_r + \ldots, \quad (22)$$

Where

$$\Upsilon_{ir} = \frac{\delta c_i(\alpha)}{\delta\alpha_r}\bigg|_{\alpha = \alpha_o} \quad (23)$$

are the unknown sensitivity vectors of the ith canonical basis axis $c_i(\alpha)$ with respect to the variation $\Delta\alpha_r$ of the r[th] parameter α_r, and

$$\lambda_i(\alpha) = \lambda_i(\alpha_o) + \sum_{r=1}^{m} \beta_{ir} \Delta\alpha_r + \ldots, \qquad (24)$$

Where

$$\beta_{ir} = \left.\frac{\delta\lambda_i(\alpha)}{\delta\alpha_r}\right|_{\alpha=\alpha_o} \qquad (25)$$

are the sensitivity functions of the ith eigenvalue $\lambda_i(\alpha)$ with respect to the variation $\Delta\alpha_r$ of α_r.

Since (22) and (24) describe the eigenvectors and eigenvalues of the perturbed matrix $W(s,\alpha)$ (11), it can be written

$$w(s)\,[R(\alpha_o) + \sum_{r=1}^{m} U_r^R \Delta\alpha_r + \ldots.] \times [c_i(\alpha_o) + \sum_{r=1}^{m} \Upsilon_{ir}\,\Delta\alpha_r + \ldots.] = w(s)\,[\lambda_i(\alpha_o) + \sum_{r=1}^{m} \beta_{ir}\,\Delta\alpha_r + \ldots] \times [c_i(\alpha_o) + \sum_{r=1}^{m} \Upsilon_{ir}\,\Delta\alpha_r + \ldots.] \qquad (26)$$

$$i = 1, 2, \ldots, N,$$

Where $\times$ is the sign of multiplication.

Performing the multiplication and comparing the first-order terms with respect to the corresponding perturbations $\Delta\alpha_r$ on both sides of (26), yields the following system of $m \cdot N$ equations:

$$R(\alpha_o)\Upsilon_{ir} + U_r^R c_i(\alpha_o) = \lambda_i(\alpha_o)\Upsilon_{ir} + \beta_{ir} c_i(\alpha_o) \qquad (27)$$

$$i = 1, 2, \ldots, N, \quad r = 1, 2, \ldots, m.$$

Taking the scalar product of both sides of (27) with the vector $c_i^+(\alpha_o)$ dual to the eigenvector $c_i(\alpha_o)$ of the unperturbed uniform system, and recalling that the first vector in the scalar product must be taken as the complex conjugate, yields

$$< R(\alpha_o)\Upsilon_{ir}, c_i^+(\alpha_o) > + < U_r^R c_i(\alpha_o), c_i^+(\alpha_o) > = \tilde{\lambda}_i(\alpha_o) < \Upsilon_{ir}, c_i^+(\alpha_o) > + \tilde{\beta}_{ir} \qquad (28)$$

$$i = 1, 2, \ldots, N; r = 1, 2, \ldots, m.$$

It should be noted that the matrix $R(\alpha_o)$ in (19) has the following canonical representation:

$$R(\alpha_o) = C(\alpha_o) diag\{\lambda_i(\alpha_o)\} C^{-1}(\alpha_o) \qquad (29)$$

Conjugation of this matrix yields

$$R^*(\alpha_o) = (C^{-1}(\alpha_o))^* diag\{\tilde{\lambda}_i(\alpha_o)\} C^*(\alpha_o) \qquad (30)$$

From here, the eigenvalue $\tilde{\lambda}_i(\alpha_o)$ of $R^*(\alpha_o)$ corresponds to the dual eigenvector $c_i^+(\alpha_o)$. Therefore,

$$< R(\alpha_o)\Upsilon_{ir}, c_i^+(\alpha_o) > = < \Upsilon_{ir}, R^*(\alpha_o) c_i^+(\alpha_o) > = \tilde{\lambda}_i(\alpha_o) < \Upsilon_{ir}, c_i^+(\alpha_o) > \qquad (31)$$

Substitution of (31) into (28) and conjugation of both sides of (28) yield the final expression for the sensitivity functions β_{ir}:

$$\beta_{ir} = < c_i^+(\alpha_o), U_r^R c_i(\alpha_o) > \qquad (32)$$

$$i = 1, 2, \ldots, N, \quad r = 1, 2, \ldots, m.$$

For a fixed parameter index r, the right-hand sides of (32) represent (for $i = 1, 2, \ldots, N$) the diagonal elements of the matrix U_r^R (21), expressed in the basis composed of the vectors $c_i(\alpha_o)$. Hence, the sensitivity functions β_{ir} of eigenvalues $\lambda_i(\alpha)$ with respect to small perturbations of the rth parameter α_r are equal to the diagonal entries of the sensitivity matrix U_r^R (21), evaluated in the canonical basis of the unperturbed uniform system.

Turn now to the calculation of the sensitivity vectors Υ_{ir} (23) of the canonical basis axes of the open-loop system. Multiplying both sides of (27) by the dual vectors $c_k^+(\alpha_o)$ ($k = 1, 2, \ldots, N$; $k \neq i$) and carrying out straightforward transformations, one obtains

$$< c_k^+(\alpha_o), \Upsilon_{ir} > = \frac{< c_k^+(\alpha_o), U_r^R c_i(\alpha_o) >}{[\lambda_i(\alpha_o) - \lambda_k(\alpha_o)]} \qquad (33)$$

$$k = 1, 2, \ldots, N; \quad k \neq i, \quad r = 1, 2, \ldots, m,$$

Where $< c_k^+(\alpha_o), U_r^R c_i(\alpha_o) > = n_{ki}^r$ are the elements of the ith column (except for n_{ii}^r) of the matrix U_r^R (21), represented in the canonical basis of the unperturbed system. The scalar products on the left-hand side of (33) are the coordinates of the vector Υ_{ir} along the kth axis $c_k(\alpha_o)$ of the initial canonical basis. By invoking the normalizing condition for the perturbed eigenvector $c_i(\alpha_o)$ (22) [17], it follows that the coordinate of Υ_{ir} along its own axis $c_i(\alpha_o)$ is equal to zero.

Therefore,

$$\Upsilon_{ir} = \sum_{\substack{k=1 \\ k \neq i}}^{N} \frac{n_{ki}^r}{[\lambda_i(\alpha_o) - \lambda_k(\alpha_o)]} c_k(\alpha_o) \qquad (34)$$

$$i = 1, 2, \ldots, N; \quad r = 1, 2, \ldots, m.$$

Accordingly, expressions (32)-(34) fully determine the sensitivity functions β_{ir} of the "factors" $\lambda_i(\alpha)$ of the CTFs $q_i(s,\alpha) = \lambda_i(\alpha) w(s)$ of the open-loop uniform system, as well as the sensitivity vectors Υ_{ir} of the canonical axes $c_i(\alpha)$

with respect to small perturbations of the parameters α_r of the matrix $R(\alpha)$. These expressions show that, in order to evaluate β_{ir} and Υ_{ir}, it is sufficient to determine the representation

$$N_r = C^{-1}(\alpha_o) U_r^R C(\alpha_o) \tag{35}$$

$$r = 1, 2, \dots, m,$$

of the sensitivity matrices U_r^R (21) in the canonical basis of the unperturbed uniform system. The diagonal elements n_{ii}^r of the matrix N_r (35) coincide with the sensitivity functions β_{ir} (25), whereas the off-diagonal elements n_{ik}^r, after division by the corresponding eigenvalue differences, give the coordinates of the sensitivity vectors Υ_{ir} (23) in the same basis.

3.2. Closed-Loop Uniform System

Consider now, following [1], the sensitivity functions of closed-loop uniform systems and their relationship to the corresponding sensitivity characteristics of the open-loop systems. For brevity, only the case of the sensitivity transfer matrix $S(s)$ (4) is discussed. The complementary transfer matrix $T(s)$ (3) can be analyzed analogously.

3.2.1. Sensitivity of the CTFs of Closed-Loop Uniform

systems with respect to parameter variations of the transfer functions $w(s, \alpha_i)$

In this case, the canonical representation of the sensitivity transfer matrix $S(s, \alpha)$ (4) via the similarity transformation is given by

$$S(s, \alpha) = C diag\{S_i(s, \alpha_i)\} C^{-1} \tag{36}$$

Where

$$S_i(s, \alpha) = \frac{1}{1 + \lambda_i w(s, \alpha_i)} \tag{37}$$

$i = 1, 2, \dots, N$, are the CTFs of $S(s, \alpha)$.

Since the modal matrix C of uniform systems is constant, the expression (36) immediately yields the following formulas for the sensitivity of the CTFs in (37) with respect to parameter variations α_{ir} of the $w(s, \alpha_i)$:

$$S_{ir}^S(s) = \left.\frac{\delta S_i(s, \alpha_i)}{\delta \alpha_{ir}}\right|_{\alpha = \alpha_o} = \frac{1}{[1 + \lambda_i w(s, \alpha_0)]^2} \lambda_i \left.\frac{\delta w(s, \alpha_i)}{\delta \alpha_{ir}}\right|_{\alpha = \alpha_o} \tag{38}$$

or by using (16),

$$S_{ir}^S(s) = -\frac{1}{[1 + \lambda_i w(s, \alpha_0)]^2} S_{ir}^q(s) \tag{39}$$

$$i = 1, 2, \dots, N; \quad r = 1, 2, \dots, m,$$

Where,

$$S_{ir}^q(s) = \lambda_i \left.\frac{\delta w(s, \alpha_i)}{\delta \alpha_{ir}}\right|_{\alpha = \alpha_o}$$

It follows from (39) that the sensitivity functions of the CTFs of the open-loop and closed-loop uniform systems, in the case of parameter variations of the transfer functions $w(s, \alpha_i)$, are related in the same way as in the classical SISO feedback control [9].

3.2.2. Sensitivity of the Closed-Loop Uniform Systems to Variation of Elements of the Cross-Connections Matrix R

Consider the sensitivity of the transfer matrix $S(s)$ (4). Assume that, under parameter perturbations $\Delta\alpha$, the original transfer matrix $S(s, \alpha_o)$ becomes

$$S(s, \alpha) = S(s, \alpha_o) + \Delta S(s, \alpha) \tag{40}$$

Where $\Delta S(s, \alpha)$ denotes the variation of $S(s, \alpha_o)$ caused by the perturbations $\Delta\alpha$.

Let us derive an expression for $S(s, \alpha)$ assuming that the corresponding variation $\Delta W(s, \alpha)$ of the open-loop uniform system is known.

In that case, the matrix $S(s, \alpha)$ (4) can be written as

$$S(s, \alpha) = [I + w(s) R(\alpha_o) + w(s) \Delta R(\alpha)]^{-1} \tag{41}$$

It follows from (4) and (41) that the variation $\Delta W(s, \alpha)$ is equal to the difference of the inverse transfer matrices $S^{-1}(s, \alpha)$ and $S^{-1}(s, \alpha_o)$, i.e.

$$S^{-1}(s, \alpha) - S^{-1}(s, \alpha_o) = w(s) \Delta R(\alpha)$$

Multiplying this relation from the left by $S(s, \alpha_o)$ and from the right by $S(s, \alpha)$ yields

$$S(s, \alpha_o) - S(s, \alpha) = -\Delta S(s, \alpha) = w(s) S(s, \alpha_o) \Delta R(\alpha) S(s, \alpha) \tag{42}$$

from which, after substituting (40) and carrying out elementary algebraic manipulations, the following expression for the variation $\Delta S(s, \alpha)$ of the transfer matrix $S(s, \alpha_o)$ caused by the variation of the open-loop transfer matrix $\Delta W(s, \alpha)$ are obtained:

$$\Delta S(s, \alpha) = -[I + wSR(s, \alpha_o)]^{-1} wSR(s, \alpha_o) S(s, \alpha_o) \tag{43}$$

Where

$$wSR(s, \alpha_o) = w(s) S(s, \alpha_o) \Delta R(\alpha)$$

In principle, this expression is valid for any, not necessarily small, variations $\Delta W(s,\alpha)$. However, since the present analysis is restricted to the first approximation and only the terms linear with respect to $\Delta\alpha_r$ are retained, let us determine this approximation for $\Delta S(s,\alpha)$. To obtain this approximation, the inverse matrix on the right-hand side of (41) is expanded into the infinite Neumann series [18]:

$$[I + S(s,\alpha_o)w(s)\Delta R(\alpha)]^{-1} = \sum_{i=0}^{\infty}[S(s,\alpha_o)w(s)\Delta R(\alpha)]^i \qquad (44)$$

This series converges whenever $||S(s,\alpha_o)w(s)\Delta R(\alpha)|| < 1$ holds, which is always the case in practice for sufficiently small $\Delta\alpha_r$. Substituting (44) in (43) and neglecting higher-order terms yields the first-order approximation for the variation $\Delta S(s,\alpha)$:

$$\Delta S(s,\alpha) = -w(s)S(s,\alpha_o)\Delta R(\alpha)S(s,\alpha_o) \qquad (45)$$

Because the transfer matrix $S(s,\alpha)$ depends continuously on the parameters α_r, the variation $\Delta S(s,\alpha)$ can be written in the form,

$$\Delta S(s,\alpha) = \sum_{r=1}^{m} U_r^S(s)\Delta\alpha_r + \ldots, \qquad (46)$$

Where

$$U_r^S(s) = \left.\frac{\delta S(s,\alpha)}{\delta\alpha_r}\right|_{\alpha=\alpha_o} \qquad (47)$$

$$r = 1, 2, \ldots, m$$

are the first-order sensitivity matrices of the transfer matrix $S(s,\alpha)$ with respect to variations of the rth parameter α_r. Then, based on (45), (46), and (21), the following relations between the sensitivity matrices (21) and (47) of the open-loop and closed-loop MIMO system transfer matrices are obtained:

$$U_r^S(s) = -w(s)S(s,\alpha_o)U_r^R S(s,\alpha_o) \qquad (48)$$

$$r = 1, 2, \ldots, m.$$

The CTFs $S_i(s)$ of the closed-loop MIMO system have the following form:

$$S_i(s,\alpha) = \frac{1}{1 + q_i(s,\alpha)} = \frac{1}{1 + \lambda_i(\alpha)w(s)}$$

$$i = 1, 2, \ldots, N.$$

Expanding these functions in a Taylor series gives

$$S_i(s,\alpha) = S_i(s,\alpha_o) + \sum_{r=1}^{m} S_{ir}^S(s)\Delta\alpha_r + \ldots \qquad (49)$$

$$i = 1, 2, \ldots, N,$$

Where

$$S_{ir}^S(s) = \left.\frac{\delta S_i(s,\alpha)}{\delta\alpha_r}\right|_{\alpha=\alpha_o} = \left.\frac{1}{[1+\lambda_i(\alpha_0)w(s)]^2}\frac{\partial q_i(s,\alpha)}{\partial\alpha_r}\right|_{\alpha=\alpha_o} = \left.\frac{1}{[1+\lambda_i(\alpha_0)w(s)]^2}w(s)\frac{\partial\lambda_i(\alpha)}{\partial\alpha_r}\right|_{\alpha=\alpha_o} = \frac{1}{[1+\lambda_i(\alpha_0)w(s)]^2}w(s)\beta_{ir} \qquad (50)$$

$$r = 1, 2, \ldots, m$$

are the first-order sensitivity functions of the CTFs $S_i(s,\alpha)$ with respect to the variations of the rth parameter α_r.

Since the canonical bases of the open-loop and closed-loop MIMO systems coincide, the expansion (22) is also valid for the axes $c_i(s,\alpha)$ of the closed-loop system, provided that the corresponding closed-loop notation $\Upsilon_{ir}^S(s)$ is introduced instead of $\Upsilon_{ir}(s)$. Proceeding as in the analysis of the open-loop uniform system, it follows that the evaluation of $S_{ir}^S(s)$ and $\Upsilon_{ir}^S(s)$ requires the representation of the matrices $U_r^S(s)$ (47) in the canonical basis of the original uniform system. That representation has the form

$$M_r(s) = C^{-1}(\alpha_o)U_r^S(s)C(\alpha_o) \qquad (51)$$

$$r = 1, 2, \ldots, m.$$

The diagonal elements $m_{ii}^r(s)$ of the matrix $M_r(s)$ coincide with the sensitivity functions $S_{ir}^S(s)$ (50), whereas the off-diagonal elements $m_{ki}^r(s)$, after division by the differences $\Delta = [S_i(s,\alpha_o) - S_k(s,\alpha_o)]$ $(i,k = 1, 2, \ldots, N, k \neq i)$, yield the sensitivity vectors $\Upsilon_{ir}^S(s)$ of the ith canonical basis axis of the closed-loop uniform system.

Expressions (48) and (51) make it possible to establish the relationship between sensitivity functions of the open-loop and closed-loop uniform systems, i.e., to determine the effect of feedback on the system sensitivity.

Substituting (48) into (51) and using the canonical representation of the transfer matrix $S(s,\alpha_o)$ (8), yields, after straightforward transformations

$$M_r(s) = -w(s)DiagS_i(s,\alpha_0)N_r DiagS_i(s,\alpha_0) \qquad (52)$$

$$r = 1, 2, \ldots, m,$$

Where

$$DiagS_i(s,\alpha_0) = diag\left\{\frac{1}{1+\lambda_i(\alpha_0)w(s)}\right\}$$

Taking the diagonal elements of the matrix on the right-hand side results in the same expression as in (50):

$$S_{ir}^{S}(s) = \frac{1}{[1+\lambda_i(\alpha_0)w(s)]^2} w(s)\beta_{ir} = \frac{1}{[1+\lambda_i(\alpha_0)w(s)]^2} S_{ir}^{q}(s) \tag{53}$$

$$i = 1, 2, \dots, N; \quad r = 1, 2, \dots, m.$$

Thus, Expressions (53) relate the sensitivity functions of the closed-loop and open-loop SISO characteristic systems and are completely analogous to those well known in classical control theory [9]. If $s = j\omega$ is assumed, then (53) shows that, at frequencies ω for which the condition $|q_i(j\omega, \alpha_o)| >> 1$ holds, typically in the low-frequency range, the introduction of feedback reduces the sensitivity of the CTFs to parameter variations. In the high-frequency range, where $|q_i(j\omega, \alpha_o)| \approx 0$ holds, the sensitivities of the closed-loop and open-loop CTFs are approximately identical. Finally, in the resonance region, where the condition $|S_i(j\omega, \alpha_o)| >> 1$ is satisfied, the feedback increases the sensitivity of the CTFs.

The off-diagonal elements $m_{ki}^{r}(s)$ of the matrix $M_r(s)$ (52) have the form

$$m_{iki}^{r}(s) = -w(s)\frac{n_{ik}^{r}}{[1+q_i(s,\alpha_o)][1+q_k(s,\alpha_o)]} \tag{54}$$

$$i,k = 1, 2, \dots, N; \quad k \neq i.$$

Dividing both sides of (54) by the differences $[S_i(s, \alpha_o) - S_k(s,\alpha_o)]$ yields the following equalities:

$$\frac{m_{ki}^{r}(s)}{S_i(s,\alpha_o) - S_k(s,\alpha_o)} = \frac{n_{ki}^{r}}{\lambda_i(\alpha_o) - \lambda_k(\alpha_o)} \tag{55}$$

$$i,k = 1, 2, \dots, N; \quad k \neq i.$$

As discussed above, the left-hand terms in (55) are the coordinates of the sensitivity vectors $\Upsilon_{ir}^{S}(s)$ of the ith canonical axis of the closed-loop uniform system, while the right-hand terms are the coordinates of the sensitivity vectors $\Upsilon_{ir}(s)$ of the corresponding, in fact identical, axis of the open-loop system. Moreover, all these coordinates are expressed with respect to the same canonical basis of the unperturbed uniform system. It follows that the vectors $\Upsilon_{ir}^{S}(s)$ and $\Upsilon_{ir}(s)$ are equal. Therefore, the introduction of feedback does not affect the sensitivity of canonical basis axes of the uniform system to small perturbations of parameters.

Finally, it should be emphasized that the results obtained in the last part of this section are more general in scope and remain applicable when the parameters of both the transfer function w(s) and the cross-connection matrix R are perturbed simultaneously.

4. Example: Control System of Space Telescope

The effectiveness of ground-based optical telescopes is largely limited by Earth's atmosphere [19]. The most radical way to avoid this issue is placing telescopes in space [20]. Figure 2 shows an astronomical telescope mounted on the satellite in a three-axis Cardan gimbal. The standard approach for pointing such telescopes relies on two bright reference stars (often called guide stars) selected near the target object. In the configuration shown in Figure 2, two star sensors (or error sensors) are mounted on the telescope in two-axis gimbals (Figure 3). The first sensor is directed at the first reference star, and the second sensor is directed toward the second star (the relative positions of the reference stars with respect to the target object are obtained from specialized Guide Star Catalogs [21]).

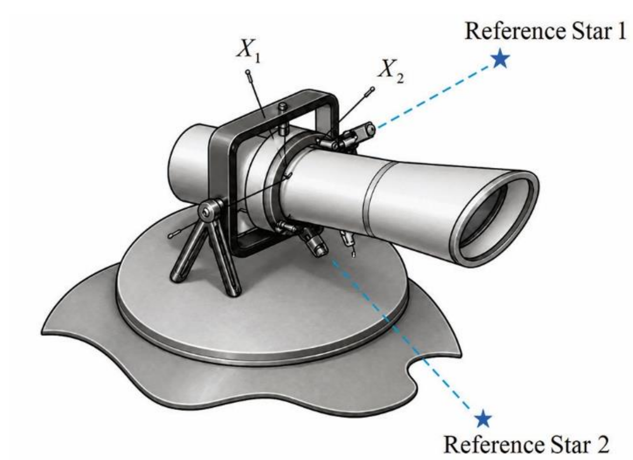


Fig. 2 Satellite-borne astronomical telescope

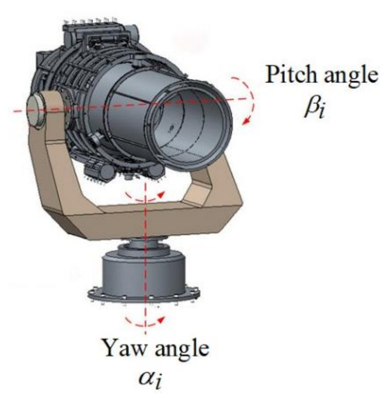
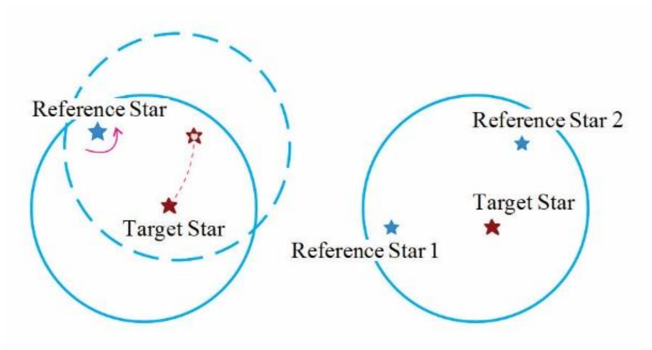


Fig. 3 Star sensor in a two-axis gimbal (*i* = 1,2)

Figure 4 illustrates the requirement for two reference stars. Each star sensor provides measurements about two orthogonal axes (pitch and yaw) and does not detect rotations about its optical (roll) axis, or about line of sight. If only one reference star is used (e.g., Star 1 in Figure 2), rotation of the entire assembly (telescope and sensor) about the line of sight generates no error signal, causing the target image in the focal plane to drift (Figure 4 (a)).



(a) (b)

Fig. 4 Guidance geometry using: (a) One reference star, and (b) Two reference stars.

Incorporating a second star sensor resolves this ambiguity by providing two additional measurement constraints (Figure 4 (b)). In this indirect guidance architecture, the sensor measurement axes do not coincide with the axes along which the servomotors are aligned (Figure 2). As a result, kinematic cross-coupling emerges between the individual channels of the system. An expanded block diagram of the transversal channels of the indirect guidance system is shown in Figure 5, where α_1 and α_2 are the sensors yaw angles,

$$R = \begin{bmatrix} \cos\alpha_1 & \sin\alpha_1 \\ -\sin\alpha_2 & \cos\alpha_2 \end{bmatrix} \tag{56}$$

is the numerical matrix of cross-connections, and $w(s)$ is the transfer function of identical individual channels.

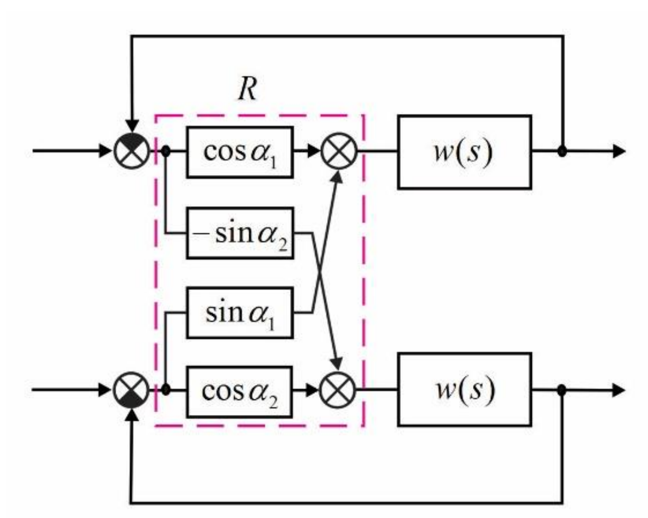


Fig. 5 Expanded block diagram of indirect guidance system (Transversal channels)

As seen from Figure 5, the transversal channels of the indirect guidance system constitute a uniform multivariable control system (the control loop about the telescope optical axis is omitted not to encumber the presentation). Each individual channel transfer function $w(s)$ is given by:

$$w(s) = Plant(s) \cdot Comp(s) \tag{57}$$

where $Plant(s)$ is the plant transfer function, incorporating the dynamics of the DC motors and star sensors:

$$Plant(s) = \frac{k}{s(s+25)(s+400)(s+500)}, \quad k = 7.5 \cdot 10^8 \tag{58}$$

and $Comp(s)$ is the transfer function of the lead-lag compensator:

$$Comp(s) = \frac{0.8(s+3)(s+25)}{(s+0.33)(s+400)} \tag{59}$$

The eigenvalues λ_1 and λ_2 of the matrix R (56) can be expressed analytically as:

$$\lambda_{1,2} = \frac{\cos\alpha_1 + \cos\alpha_2}{2} \pm \sqrt{\frac{(\cos\alpha_1 + \cos\alpha_2)^2}{4} - \cos(\alpha_1 - \alpha_2)} \tag{60}$$

To analyze the sensitivity of the indirect guidance system, three distinct operational scenarios are evaluated based on the signs and the values of angles α_1 and α_2 in (56) and (60).

Scenario 1: $\alpha_1 = 40°, \quad \alpha_2 = 20°$.

The relative geometric arrangement of the sensors measurements axes X_1, X_2 and the telescope motors' axes Y_M, X_M is shown in Figure 6, where γ represents the angle between star sensors measurement axes.

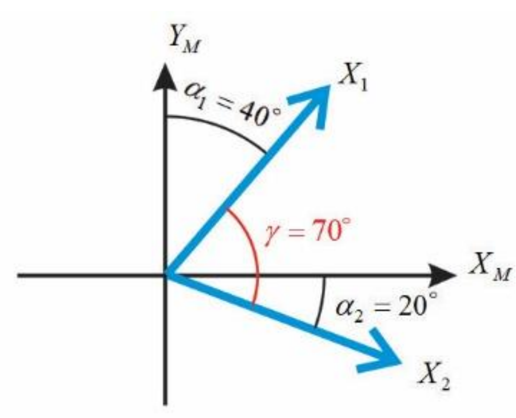


Fig. 6 Measurements axes (blue lines) and motors axes for Scenario 1

In this case, matrix R is given by (61):

$$R = \begin{bmatrix} 0.766 & 0.6428 \\ -0.342 & 0.9397 \end{bmatrix} \tag{61}$$

yielding complex conjugate-conjugate eigenvalues λ_1 and λ_2:

$$\lambda_{1,2} = 0.8529 \pm j0.4608 \tag{62}$$

The corresponding modal matrix C, formed by the eigenvectors c_i of R (61), is

$$C = \begin{bmatrix} 0.8079 & 0.8079 \\ 0.1091 + j0.5791 & 0.1091 - j0.5791 \end{bmatrix} \qquad (63)$$

The condition number of the matrix C (63), which is a measure of non-orthogonality of axes, X_1, X_2 is equal to

$$||C|| \cdot ||C^{-1}|| = 1.4555 \qquad (64)$$

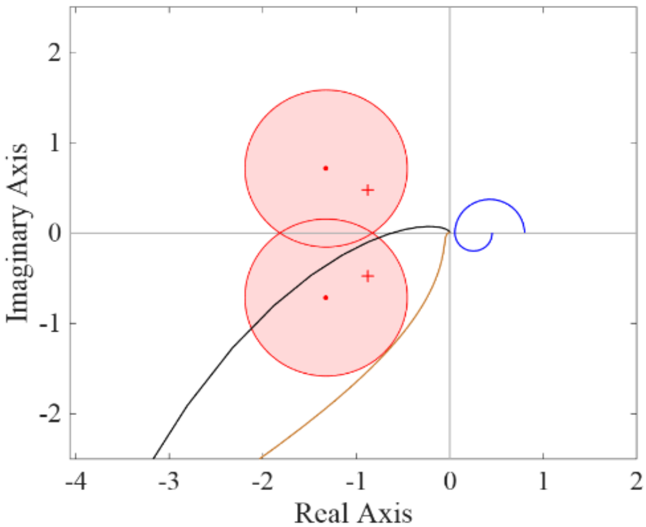


Fig. 7 Nyquist plots of the CTFs for scenario 1

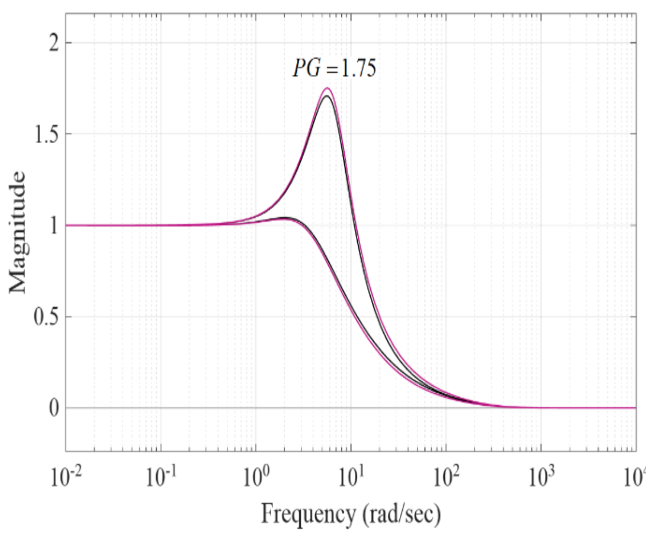


Fig. 8 Generalized frequency responses for Scenario 1

The Nyquist plots, generalized frequency responses, and step response characteristics of the guidance system for Scenario 1 are shown in Figures 7 - 9. The black line in Figure 7 corresponds to $Plant(s)$, blue line to $Comp(s)$, brown line to transfer function $w(s)$ (57), and the red circles are forbidden regions corresponding to peak gain PG = 1.75 (H_∞-norm) in Figure 8.

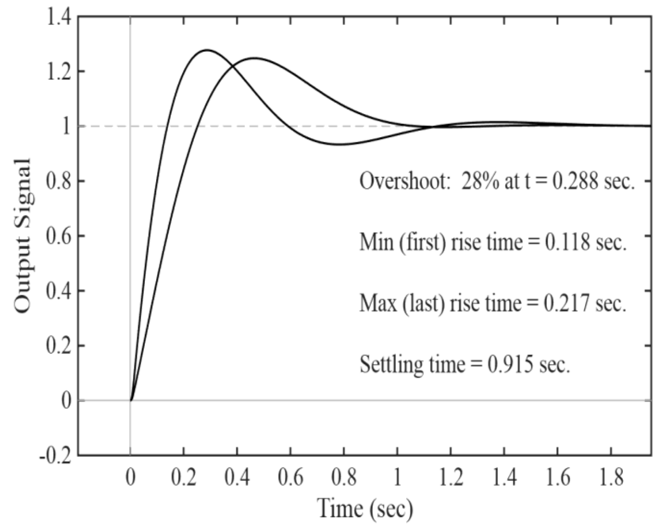


Fig. 9 Step responses for Scenario 1

Determine now, using the results presented in Section 3, the sensitivities β_{11}, β_{21} and $\Upsilon_{11}, \Upsilon_{21}$ of the factors λ_1 λ_2 in (65) and the canonical basis axes c_1, c_2 with respect to small perturbations of angle α_1 in R (66). The calculations yield

$$\begin{aligned} \beta_{11} &= \frac{\delta\lambda_1(\alpha_1,\alpha_2)}{\delta\alpha_1} = -0.3214 + j0.2237, \\ \beta_{21} &= \frac{\delta\lambda_2(\alpha_1,\alpha_2)}{\delta\alpha_1} = -0.3214 - j0.2237 \end{aligned} \qquad (65)$$

$$\begin{aligned} \Upsilon_{11} &= \frac{\delta c_1(\alpha_1,\alpha_2)}{\delta\alpha_1} = \begin{bmatrix} 0.2853 + j0.1910 \\ 0.1755 - j0.1787 \end{bmatrix}, \\ \Upsilon_{21} &= \frac{\delta c_2(\alpha_1,\alpha_2)}{\delta\alpha_1} = \begin{bmatrix} 0.2853 - j0.1910 \\ 0.1755 + j0.1787 \end{bmatrix} \end{aligned} \qquad (66)$$

The magnitude plots $|S_{11}^S(j\omega)|$ and $|S_{21}^S(j\omega)|$ of the closed-loop sensitivity functions with respect to small variations in the plant gain k (58) for the first channel are illustrated in Figure 10.

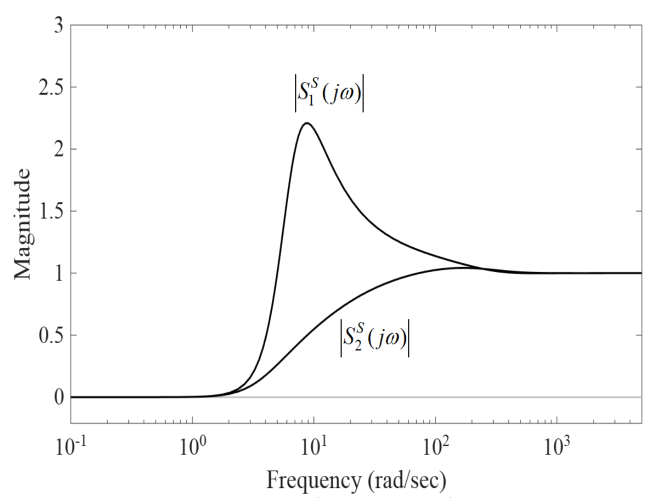


Fig. 10 Sensitivity magnitudes $|S_{11}^S(j\omega)|$ and $|S_{21}^S(j\omega)|$ for Scenario 1

Sensitivities of the closed-loop guidance system to small perturbations of the angle α_2 and gain k in the second channel are determined analogously.

Scenario 2: $\alpha_1 = 40°, \quad \alpha_2 = -20°$

For this scenario, the alignment of the sensors' measurement axes relative to motors axes is shown in Figure 11. The cross-connection matrix R (67) takes the following form:

$$R = \begin{bmatrix} 0.766 & 0.6428 \\ 0.342 & 0.9397 \end{bmatrix} \tag{67}$$

yielding real-valued eigenvalues λ_1 and λ_2:

$$\lambda_1 = 0.376, \quad \lambda_2 = 1.3297 \tag{68}$$

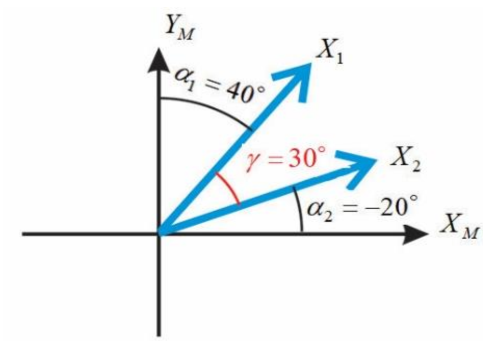


Fig. 11 Measurements and motors axes for Scenario 2

Correspondingly, the modal matrix C is real-valued:

$$C = \begin{bmatrix} -0.8549 & -0.7519 \\ 0.5187 & -0.6593 \end{bmatrix} \tag{69}$$

with a condition number $||C|| \cdot ||C^{-1}||$ equal to

$$||C|| \cdot ||C^{-1}|| = 1.364 \tag{70}$$

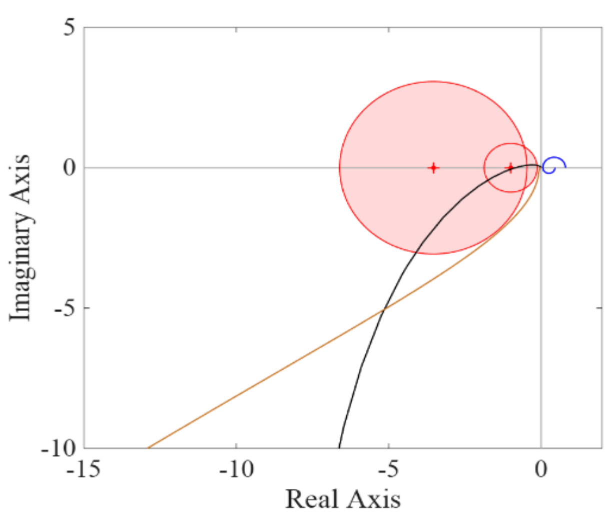


Fig. 12 Nyquist plots of the CTFs for Scenario 2

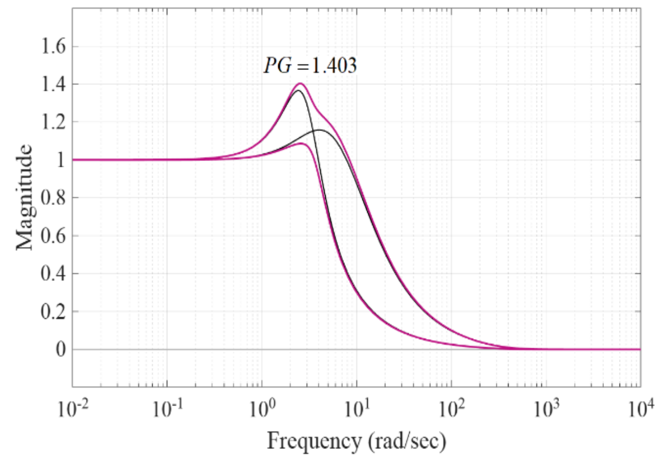


Fig. 13 Generalized frequency responses for Scenario 2

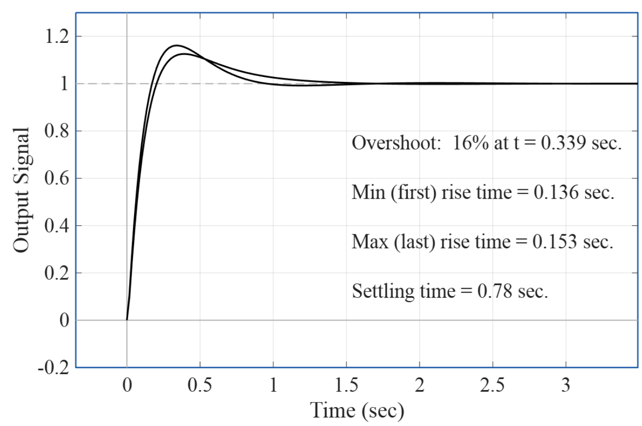


Fig. 14 Step responses for Scenario 2

The Nyquist plots, generalized frequency responses, and step response characteristics of the guidance system for Scenario 2 are shown in Figures 12-14.

The sensitivities β_{11}, β_{21} and $\Upsilon_{11}, \Upsilon_{21}$ of the eigenvalues λ_1, λ_2 and the canonical basis axes c_1, c_2 with respect to small perturbations in angle α_1 are given by the following expressions (note that all sensitivities are real-valued):

$$\beta_{11} = \frac{\delta\lambda_1(\alpha_1,\alpha_2)}{\delta\alpha_1} = -0.6546$$

$$\beta_{21} = \frac{\delta\lambda_2(\alpha_1,\alpha_2)}{\delta\alpha_1} = 0.0118 \tag{71}$$

$$\Upsilon_{11} = \frac{\delta c_1(\alpha_1,\alpha_2)}{\delta\alpha_1} = \begin{bmatrix} -0.4061 \\ -0.3561 \end{bmatrix}$$

$$\Upsilon_{21} = \frac{\delta c_2(\alpha_1,\alpha_2)}{\delta\alpha_1} = \begin{bmatrix} -0.0135 \\ 0.0082 \end{bmatrix} \tag{72}$$

The magnitude plots $|S_{11}^S(j\omega)|$ and $|S_{21}^S(j\omega)|$ of the closed-loop sensitivity functions with respect to small perturbations of the plant gain k (58) in the first channel are shown in Figure 15.

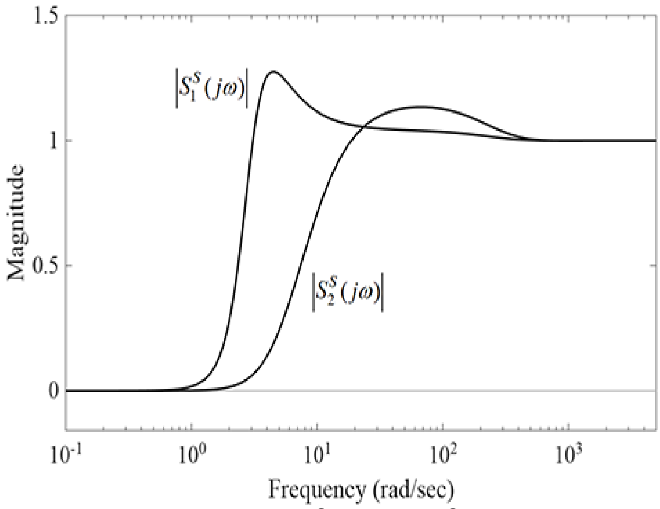


Fig. 15 Sensitivity magnitudes $|S_{11}^S(j\omega)|$ and $|S_{21}^S(j\omega)|$ for Scenario 2

Scenario 3: $\alpha_1 = 40°$, $\alpha_2 = 40°$

This scenario exhibits specific structural and geometric symmetry. Since $\alpha_1 = \alpha_2$the measurement axes of the star sensors are mutually orthogonal and the coordinate frame X_1X_2 is obtained from the orthogonal frame Y_MX_M by a pure rotation by an angle 40° (Figure 16).

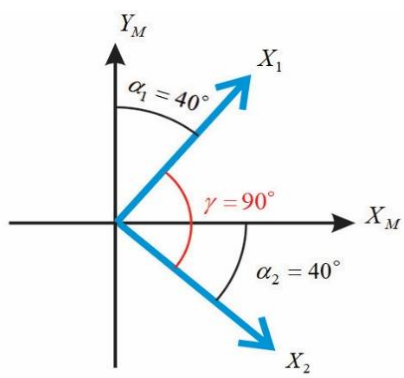


Fig. 16 Measurements and motors axes for Scenario 3

As a result, the matrix Rhas the form.

$$R = \begin{bmatrix} 0.7660 & 0.6428 \\ -0.6428 & 0.7660 \end{bmatrix} \tag{73}$$

and is orthogonal, that is

$$R^{-1} = R^T \tag{74}$$

The complex-conjugate eigenvalues λ_1 and λ_2 of that matrix:

$$\lambda_{1,2} = 0.766 \pm j0.6428 = exp\{\pm 40°\} \tag{75}$$

lie on the unit circle in the complex plane, and the model matrix

$$C = \begin{bmatrix} j0.7071 & -j0.7071 \\ 0.7071 & 0.7071 \end{bmatrix} \tag{76}$$

is orthogonal and has a unit condition number

$$||C|| \cdot ||C^{-1}|| = 1.0 \tag{77}$$

The Nyquist plots, generalized frequency responses, and step responses of the guidance system for Scenario 3 are shown in Figures 17 - 19.

The sensitivities function β_{11}, β_{21} of the eigenvalues λ_1, λ_2 and sensitivity vectors $\Upsilon_{11}, \Upsilon_{21}$ of the canonical basis axes c_1, c_2 with respect to small perturbations of angle α_1 are given by the following expressions:

$$\begin{aligned} \beta_{11} &= \frac{\delta\lambda_1(\alpha_1,\alpha_2)}{\delta\alpha_1} = -0.3214 + j0.383, \\ \beta_{21} &= \frac{\delta\lambda_2(\alpha_1,\alpha_2)}{\delta\alpha_1} = -0.3214 - j0.383 \end{aligned} \tag{78}$$

$$\begin{aligned} \Upsilon_{11} &= \frac{\delta c_1(\alpha_1,\alpha_2)}{\delta\alpha_1} = \begin{bmatrix} 0.1768 - j0.2107 \\ -0.2107 - j0.1768 \end{bmatrix} \\ \Upsilon_{21} &= \frac{\delta c_2(\alpha_1,\alpha_2)}{\delta\alpha_1} = \begin{bmatrix} 0.1768 + j0.2107 \\ -0.2107 + j0.1768 \end{bmatrix} \end{aligned} \tag{79}$$

The magnitude plots $|S_{11}^S(j\omega)|$ and $|S_{21}^S(j\omega)|$ of the closed-loop sensitivity functions with respect to small perturbations of the plant gain k (58) in the first channel are shown in Figure 20.

A comparative analysis of the dynamic performance and sensitivity characteristics of the guidance system across all three configurations indicates that Scenario 2, for which the eigenvalues and eigenvectors of the matrix R (56) are real-valued, provides the most favorable performance.

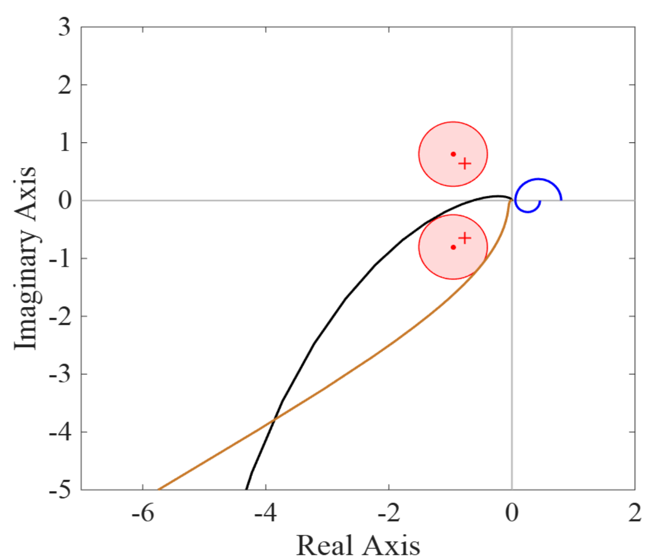


Fig. 17 Nyquist plots of the CTFs for Scenario 3

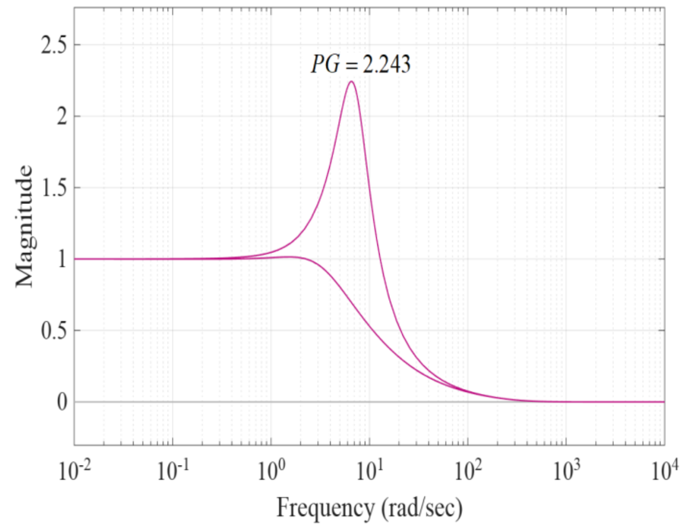


Fig. 18 Generalized frequency responses for Scenario 3

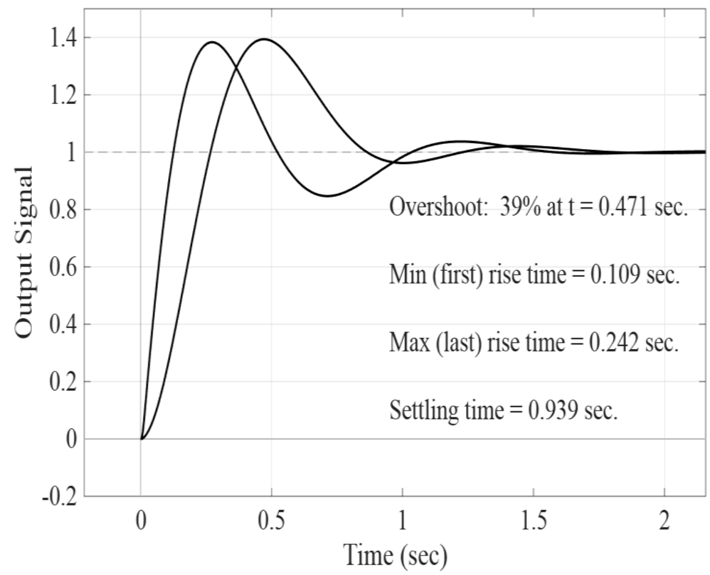


Fig. 19 Step responses for Scenario 3

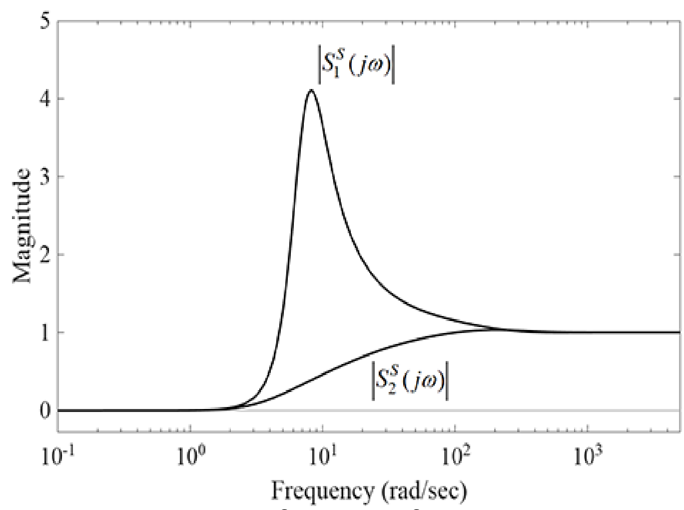


Fig. 20 Magnitudes $|S_{11}^S(j\omega)|$ and $|S_{21}^S(j\omega)|$ for Scenario 3

5. Conclusion

In this paper, the sensitivity of uniform multivariable control systems was investigated using the CTFs method. The presented results extend the sensitivity analysis of the CTFs and canonical basis axes originally developed for the general type multivariable control systems [3] to the class of uniform systems.

Structurally, uniform systems consist of two decoupled matrix operators: a scalar transfer matrix describing the identical independent channels and a constant numerical matrix describing rigid cross-connections. Parameter perturbations originating in these two blocks produce fundamentally distinct effects on system CTFs and canonical axes. In this work, these perturbation mechanisms were treated independently, establishing closed-form sensitivity expressions for scenarios where perturbations are concentrated either entirely within the channel dynamics or within the interconnection numerical matrix.

The practical application of the results is demonstrated through the sensitivity analysis of an indirect guidance system of an astronomical telescope mounted on the satellite in the Cardan gimbal.

Funding

This research received no external funding

Conflicts of Interest

The authors declare no conflicts of interest.

References

[1] Sigurd Skogestad, and Ian Postlethwaite, *Multivariable Feedback Control: Analysis and Design*, 2nd ed., John Wiley and sons, 2005. [Google Scholar] [Publisher Link]

[2] J.M. Maciejowski, "Multivariable Feedback Design," *Cambridge University Press*, vol. 8, no. 2, 2017. [Publisher Link]

[3] Oleg Gasparyan et al., "On Sensitivity of Characteristic Transfer Functions of Multivariable Control Systems," *Automation*, vol. 7, no. 3, pp. 1-24, 2026. [CrossRef] [Google Scholar] [Publisher Link]

[4] Ian Postlethwaite, and Alistair G.J. MacFarlane, *A Complex Variable Approach to the Analysis of Linear Multivariable Feedback Systems*, Springer Berlin, Heidelberg, vol. 12, 1979. [CrossRef] [Google Scholar] [Publisher Link]

[5] A.G.J. Macfarlane, and J.J. Belletrutti, "The Characteristic Locus Design Method," *Automatica*, vol. 9, no. 5, pp. 575-588, 1973. [CrossRef] [Google Scholar] [Publisher Link]

[6] A.G.J. MacFarlane, and I. Postlethwaite, "Characteristic Frequency Functions and Characteristic Gain Functions," *International Journal of Control*, vol. 26, no. 2, pp. 265-278, 1977. [CrossRef] [Google Scholar] [Publisher Link]

[7] A.G.J. MacFarlane, and B. Kouvaritakis, "A Design Technique for Linear Multivariable Feedback Systems," *International Journal of Control*, vol. 25, no. 6, pp. 837-874, 1977. [CrossRef] [Google Scholar] [Publisher Link]

[8] A.G.J. MacFarlane, "Commutative Controller: A New Technique for the Design of Multivariable Control Systems," *Electronics Letters*, vol. 6, no. 5, pp. 121-123, 1970. [CrossRef] [Google Scholar] [Publisher Link]

[9] Richard C. Dorf, and Robert H. Bishop, *Modern Control Systems*, 14th ed., Pearson, 2022. [Google Scholar] [Publisher Link]

[10] Ian Postlethwaite, "Sensitivity of the Characteristic Gain Loci," *Automatica*, vol. 18, no. 6, pp. 709-712, 1982. [CrossRef] [Google Scholar] [Publisher Link]

[11] M.V. Moreira, and J.C. Basilio, "Characteristic Locus Method Robustness Improvement through Optimal Static Normalizing Pre-Compensation," *International Journal of Robust and Nonlinear Control*, vol. 20, no. 4, pp. 371-386, 2010. [CrossRef] [Google Scholar] [Publisher Link]

[12] R. Tomovic, M. Vulkobratovic, and W.V. Loscutoff, "General Sensitivity Theory," *Journal of Dynamic Systems, Measurement, and Control*, vol. 95, no. 2, pp. 242-243, 1973. [CrossRef] [Google Scholar] [Publisher Link]

[13] Mansour Eslami, *Theory of Sensitivity in Dynamic Systems: An Introduction*, 1st ed., Springer Berlin, Heidelberg, 1994. [CrossRef] [Google Scholar] [Publisher Link]

[14] Efim Rozenwasser, and Rafael Yusupov, *Sensitivity of Automatic Control Systems*, 1st ed., Taylor and Francis Group, 1999. [CrossRef] [Google Scholar] [Publisher Link]

[15] Jose B. Cruz, and W. Perkins, "A New Approach to the Sensitivity Problem in Multivariable Feedback System Design," *IEEE Transactions on Automatic Control*, vol. 9, no. 3, pp. 216-223, 1964. [CrossRef] [Google Scholar] [Publisher Link]

[16] B. Morgan, "Sensitivity Analysis and Synthesis of Multivariable Systems," *IEEE Transactions on Automatic Control*, vol. 11, no. 3, pp. 506-512, 1966. [CrossRef] [Google Scholar] [Publisher Link]

[17] Izrail′ Moiseevich Gelfand, *Lectures on Linear Algebra*, Courier Corporation, 1989. [Google Scholar]

[18] Tosio Kato, *Perturbation Theory for Linear Operators*, Springer Berlin, Heidelberg, vol. 132, 1995. [CrossRef] [Google Scholar] [Publisher Link]

[19] Jacques M. Beckers, "Adaptive Optics for Astronomy: Principles, Performance, and Applications," *Annual Review of Astronomy and Astrophysics*, vol. 31, pp. 13-62, 1993. [CrossRef] [Google Scholar] [Publisher Link]

[20] G. Fritz Benedict et al., "Astrometry with Hubble Space Telescope Fine Guidance Sensors-A Review," *Publications of the Astronomical Society of the Pacific*, vol. 129, no. 971, pp. 1-16, 2017. [CrossRef] [Google Scholar] [Publisher Link]
[21] Barry M. Lasker et al., "The Second-Generation Guide Star Catalog: Description and Properties," *The Astronomical Journal*, vol. 136, no. 2, pp. 735-766, 2008. [CrossRef] [Google Scholar] [Publisher Link]